ECE 313: Problem Set 9: Solutions
Gaussian Random Variable, Functions of a Random Variable

1. **[Gaussian CDF]**

   (a) The derivative of $\exp(-u^2/2)$ is $-u\exp(-u^2/2)$. Hence

   $$E[|\mathbb{X}|] = \int_{-\infty}^{\infty} \frac{|u|}{\sqrt{2\pi}} \exp(-u^2/2)du = 2\int_{0}^{\infty} \frac{|u|}{\sqrt{2\pi}} \exp(-u^2/2)du = \frac{\sqrt{2}}{\sqrt{\pi}}.$$

   (b) Since $t + x > t - x > 0$ we have that $(t + x)(t - x) = t^2 - x^2 > (t - x)^2 > 0$. Hence

   $$\exp(x^2/2)Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp((-t^2 + x^2)/2)dt \le \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-(t-x)^2/2)dt = 0.5.$$

2. **[Table of Q functions]**

   (a) Let $\mathbb{X} \sim N(0.9, 0.003^2)$ denote the width in microns of the trace. For a trace to be defective we must have $\mathbb{X} < 0.9 - 0.005$ or $\mathbb{X} > 0.9 + 0.005$. The probability is $P(|\mathbb{X} - 0.9| > 0.005) = 2\Phi(-0.005/0.003) = 2\Phi(-1.666\cdots) = 2Q(1.666\cdots) \approx 0.095$.

   (b) We need to find the maximum value of σ such that $2Q(0.005/\sigma) \le 0.01$. Since $Q(2.575) \approx 0.005$ we get that $\sigma \approx 0.00194$.

3. **[Function of a Random Variable]**

   (a) $\mathcal{I}$ can take on values in the range $(-I_0, \infty)$.

   (b) $F_{\mathcal{I}}(v) = 0$ for $v < -I_0$. For any $v > -I_0$,
   $F_{\mathcal{I}}(v) = P\{\mathcal{I} \le v\} = P\{I_0(\exp(\mathcal{V}) - 1) \le v\} = P\{\mathcal{V} \le \ln(1 + v/I_0)\} = F_{\mathcal{V}}(\ln(1 + v/I_0))$.

   (c) For $v > -I_0$,

   $$f_{\mathcal{I}}(v) = f_{\mathcal{V}}(\ln(1 + v/I_0))\frac{1}{1 + v/I_0} \times \frac{1}{I_0} = \frac{f_{\mathcal{V}}(\ln(1 + v/I_0))}{v + I_0} = \begin{cases} \frac{I_0/2}{(v+I_0)^2}, & v \ge 0, \\ \frac{1}{2I_0}, & -I_0 < v < 0, \end{cases}.$$

   Note that the pdf has constant value $1/(2I_0)$ from $v = -I_0$ to $v = 0$.

4. []

   (a) $\mathbb{Y}$ can take on only two values in the set $\{-\alpha, \alpha\}$. Further $P\{\mathbb{Y} = -\alpha\} = P\{\mathbb{X} \le 0\} = 0.5$. It follows that $P\{\mathbb{Y} = \alpha\} = P\{\mathbb{X} > 0\} = 0.5$.

   (b) Using LOTUS

   $$\begin{aligned} \mathsf{E}[\mathbb{Z}] &= \int_{-\infty}^{0} (x+\alpha)^2 f_{\mathbb{X}}(x)\,dx + \int_{0}^{\infty} (x-\alpha)^2 f_{\mathbb{X}}(x)\,dx \\ &= \int_{-\infty}^{\infty} (x^2+\alpha^2) f_{\mathbb{X}}(x)\,dx - 2\alpha\int_{0}^{\infty} x\, f_{\mathbb{X}}(x)\,dx + 2\alpha\int_{-\infty}^{0} x\, f_{\mathbb{X}}(x)\,dx \\ &= (1+\alpha^2) - 4\alpha\int_{0}^{\infty} x\, f_{\mathbb{X}}(x)\,dx \\ &= (1+\alpha^2) - \frac{4\alpha}{\sqrt{2\pi}}\int_{0}^{\infty} x\, e^{-\frac{x^2}{2}}\,dx \\ &= (1+\alpha^2) - \frac{4\alpha}{\sqrt{2\pi}}. \end{aligned}$$

   The choice of α that minimizes $\mathsf{E}[\mathbb{Z}]$ is $\frac{2}{\sqrt{2\pi}}$.

(c) $\mathbb{W}$ takes values in $\{\pm 3, \pm 2, \pm 1, 0\}$. The probabilities are

$$\begin{aligned}
P\{\mathbb{W}=0\} &= P\{-0.5 \leq \mathbb{X} < 0.5\} = 1 - 2Q(0.5) \\
P\{\mathbb{W}=1\} &= P\{0.5 \leq \mathbb{X} < 1.5\} = Q(0.5) - Q(1.5) \\
P\{\mathbb{W}=-1\} &= P\{-1.5 \leq \mathbb{X} < -0.5\} = Q(0.5) - Q(1.5) \\
P\{\mathbb{W}=2\} &= P\{1.5 \leq \mathbb{X} < 2.5\} = Q(1.5) - Q(2.5) \\
P\{\mathbb{W}=2\} &= P\{-2.5 \leq \mathbb{X} < -1.5\} = Q(1.5) - Q(2.5) \\
P\{\mathbb{W}=3\} &= P\{2.5 \geq \mathbb{X}\} = Q(2.5) \\
P\{\mathbb{W}=-3\} &= P\{-2.5 > \mathbb{X}\} = Q(2.5).
\end{aligned}$$

5. []

(a) The average radius of the sphere is $\int_0^1 u2u\,du = \frac{2}{3}$.

(b) The average volume is $\int_0^1 \frac{4\pi u^3}{3} \cdot 2u\,du = \frac{8\pi}{15}$.

(c) The average surface area is $\int_0^1 4\pi u^2 \cdot 2u\,du = 2\pi$.

(d) The average sphere has volume $\frac{32\pi}{91}$ which is larger than the average volume. The average sphere has surface area $\frac{16\pi}{9}$ which is larger than the average surface area.

6. []

Suppose the support of $\mathbb{X}$ is $[a, b]$. Then $a+b=2$ and $(b-a)^2 = 36$. This allows us to calculate $a=-2$ and $b=4$. The pdf $f_{\mathbb{Y}}(y)$is naturally divided into two regions: $0 \leq y \leq 2$ and $4 \geq y > 2$:

$$f_{\mathbb{Y}}(y) = \begin{cases} \frac{1}{3} & 0 \leq y \leq 2 \\ \frac{1}{6} & 2 < y \leq 4 \\ 0 & y > 4. \end{cases}$$

Formal derivation is as follows: For $y < 0$, the CDF $F_{\mathbb{Y}}(y) = 0$. Further for $y > 4$, the CDF $F_{\mathbb{Y}}(y) = 1$.
For any $0 \leq y \leq 2$, the CDF $F_{\mathbb{Y}}(y) = P\{\mathbb{Y} \leq y\} = P\{-y \leq \mathbb{X} \leq y\} = \frac{y}{3}$.
For any $2 \leq y \leq 4$, the CDF $F_{\mathbb{Y}}(y) = P\{\mathbb{Y} \leq y\} = P\{\mathbb{X} \leq y\} = \frac{y+2}{6}$.
Now $f_{\mathbb{Y}}(y) = \frac{d}{dy} F_{\mathbb{Y}}(y)$ follows directly.