University of Illinois **Spring 2013**

ECE 313: Problem Set 8: Problems and Solutions
Poisson process; scaling of RVs; Erlang and Gaussian pdf; ML estimation

Due: Wednesday, March 13 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.5–3.7

1. **[Exam Grading]**
Suppose students in a class (not ECE313) take an exam. Assume the exam scores are distributed according to a Gaussian distribution with mean 75 and standard deviation 15. Letter grades are assigned to students according to one of the following grading policies.

Policy 1		Policy 2	
Scores	Grade	Ranking	Grade
85 or more	A	Top 17%	A
70 or more	B	Top 50%	B
55 or more	C	Top 83%	C
Less than 55	D	Not in top 83%	D

(a) Using Tables 6.1 and 6.2 in the lecture notes, find the percentage of students expected to receive grades of A, B, C, and D under grading policy 1. **Solution:**

$$\begin{aligned} P(A) &= Q\left(\frac{85-75}{15}\right) = 0.2514 \\ P(B) &= Q\left(\frac{70-75}{15}\right) - Q\left(\frac{85-75}{15}\right) = 0.3779 \\ P(C) &= Q\left(\frac{55-75}{15}\right) - Q\left(\frac{70-75}{15}\right) = 0.2789 \\ P(D) &= \Phi\left(\frac{55-75}{15}\right) = 0.0918 \end{aligned}$$

(b) Using the same tables, find the cut-off scores between A and B, B and C, and C and D under grading policy 2. It is not necessary that the cut-off scores must be integers. **Solution:**

$$\begin{aligned} Q\left(\frac{X_A-75}{15}\right) = 0.17 \quad &\Rightarrow \quad X_A = 75 + 0.95 \times 15 = 89.25 \\ Q\left(\frac{X_B-75}{15}\right) = 0.5 \quad &\Rightarrow \quad X_B = 75 \\ Q\left(\frac{X_C-75}{15}\right) = 0.83 \quad &\Rightarrow \quad X_C = 75 - 0.95 \times 15 = 60.75 \end{aligned}$$

2. **[Poisson process]**
Consider a process in which successes can occur at any time, at an average rate of λ events per second. The number of events occurring in any time interval is independent of the number of events occurring in any other time interval unless the intervals overlap. Define the following random variables:

- X_m is the number of successes achieved in the first m seconds, i.e., during the time interval $t \in (0, m]$.
- X_n is the number of successes achieved in the first n seconds, i.e., during the time interval $t \in (0, n]$, where $n > m$ in all parts of this problem.
- T_j is the time (in seconds) at which the j^{th} success occurs.
- T_k is the time (in seconds) at which the k^{th} success occurs, where $k > j$ in all parts of this problem.

In terms of the variables λ, m, n, j, k, etc., find the following pmfs and pdfs:

(a) $p_{X_n}(k)$ is the probability that there are k successes in the first n seconds. Write a formula for $p_{X_n}(k)$.
Solution: X_n is a Poisson random variable, with pmf

$$p_{X_n}(k) = \begin{cases} \frac{(n\lambda)^k e^{-n\lambda}}{k!} & 0 \leq k \\ 0 & \text{otherwise} \end{cases}$$

(b) $p_{X_n|X_m}(k|i)$ is the probability that there are k successes in the first n seconds, given that there are i successes in the first m seconds, where $m < n$. Write a formula for $p_{X_n|X_m}(k|i)$.
Solution: There are i successes in the interval $t \in (0, m]$, therefore there must be $k - i$ successs in the interval $t \in (m, n]$. Non-overlapping intervals of time are independent, therefore this is just another Poisson random variable, with parameter $(n - m)\lambda$:

$$p_{X_n|X_m}(k|i) = \begin{cases} \frac{((n-m)\lambda)^{k-i} e^{-(n-m)\lambda}}{(k-i)!} & 0 \leq k - i \\ 0 & \text{otherwise} \end{cases}$$

(c) $p_{X_m|X_n}(i|k)$ is the probability that there are i successes in the first m seconds, given that there are k successes in the first n seconds, where $m < n$. Write a formula for $p_{X_m|X_n}(i|k)$.
Solution: The interval $t \in (0, m]$ is a strict subset of the interval $t \in (0, n]$, so the non-overlapping intervals argument used in part (b) does not work here. Instead, notice that successes are equally likely to occur at any time: therefore, conditioned on the fact that a success happens in the interval $(0, n]$, the probability that it occurs in the interval $(0, m]$ is just m/n. Similarly, the probability that any particular set of i events happens in $(0, m]$ while another particular set of $k - i$ events happens in $(m, n]$ is $\left(\frac{m}{n}\right)^i \left(\frac{n-m}{n}\right)^{k-i}$. There are $\begin{pmatrix} k \\ i \end{pmatrix}$ ways to choose a subset of i successes out of a set of k successes, so the probability of getting any arbitrary i successes in the first m seconds is

$$p_{X_m|X_n}(i|k) = \begin{cases} \begin{pmatrix} k \\ i \end{pmatrix} \left(\frac{m}{n}\right)^i \left(\frac{n-m}{n}\right)^{k-i} & 0 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

(d) $f_{T_1}(u)$ is the probability density with which the first success happens at exactly $T_1 = u$ seconds, where $u \geq 0$ is a real-valued instance variable. Write a formula for $f_{T_1}(u)$.

Solution: This is an exponential random variable, with pdf

$$f_{T_1}(u) = \begin{cases} \lambda e^{-\lambda u} & u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(e) $f_{T_k}(v)$ is the probability density with which the k^{th} success happens at exactly $T_k = v$ seconds, where v is a real-valued instance variable. Write a formula for $f_{T_k}(v)$.

Solution: This is an Erlang variable, with pdf

$$f_{T_k}(v) = \begin{cases} \frac{\lambda^k v^{k-1} e^{-\lambda v}}{(k-1)!} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(f) $f_{T_k|T_j}(v|u)$ is the probability density with which the k^{th} success happens at $T_k = v$ seconds, given that the j^{th} success happens at $T_j = u$ seconds, where $j < k$. Write a formula for $f_{T_k|T_j}(v|u)$.

Solution: The time intervals $(0, u]$ and $(u, v]$ are non-overlapping, therefore the events that occur during these two time intervals are independent. The probability density $f_{T_k|T_j}(v|u)$ is therefore identical to the probability density of getting the $(k-j)^{\text{th}}$ success at $(v - u)$ seconds, i.e.,

$$f_{T_k|T_j}(v|u) = \begin{cases} \frac{\lambda^{k-j}(v-u)^{k-j-1} e^{-\lambda(v-v)}}{(k-j-1)!} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(g) $f_{T_j|T_k}(u|v)$ is the probability density with which the j^{th} success happens at $T_j = u$ seconds, given that the k^{th} success happened at $T_k = v$ seconds, where $j < k$. Write a formula for $f_{T_j|T_k}(u|v)$.

Solution: The time interval $(0, u]$ is a strict subset of the interval $(0, v]$, therefore the independence argument used in part (f) won't work here. There are a few equivalent solutions to this problem, but the easiest is to reason in terms of the definition of conditional probability, like this:

$$
\begin{aligned}
f_{T_j|T_k}(u|v)du &= \Pr\left\{T_j \in u \pm \frac{du}{2} | T_k = v\right\} \\
&= \frac{\Pr\left\{T_j \in u \pm \frac{du}{2}, T_k \in v \pm \frac{dv}{2}\right\}}{\Pr\left\{T_k \in v \pm \frac{dv}{2}\right\}} \\
&= \frac{\Pr\left\{T_j \in u \pm \frac{du}{2}, T_{k-j} \in (v-u) \pm \frac{dv}{2}\right\}}{\Pr\left\{T_k \in v \pm \frac{dv}{2}\right\}} \\
&= \frac{\Pr\left\{T_j \in u \pm \frac{du}{2}\right\} \Pr\left\{T_{k-j} \in (v-u) \pm \frac{dv}{2}\right\}}{\Pr\left\{T_k \in v \pm \frac{dv}{2}\right\}} \\
&= \frac{f_{T_j}(u) f_{T_{k-j}}(v-u)dudv}{f_{T_k}(v)dv} \\
&= \left(\frac{\lambda^j u^{j-1} e^{-\lambda u}}{(j-1)!}\right) \left(\frac{\lambda^{k-j}(v-u)^{k-j-1} e^{-\lambda(u-v)}}{(k-j-1)!}\right) \left(\frac{(k-1)!}{\lambda^k v^{k-1} e^{-\lambda v}}\right) \\
&= \frac{(k-1)!}{(j-1)!(k-j-1)!} \left(\frac{1}{v}\right) \left(\frac{u}{v}\right)^{j-1} \left(\frac{v-u}{v}\right)^{k-j-1} du
\end{aligned}
$$

So

$$
f_{T_j|T_k}(u|v) = \begin{cases} \frac{(k-1)!}{(j-1)!(k-j-1)!} \left(\frac{1}{v}\right) \left(\frac{u}{v}\right)^{j-1} \left(\frac{v-u}{v}\right)^{k-j-1} & 0 \le u \le v \\ 0 & \text{otherwise} \end{cases}
$$

3. **[Serial and parallel Poisson processes]**
I have a very small house; there are only 4 light bulbs in my house. Each light bulb burns out at an average rate of 0.1 burnouts per light bulb, per day. Burnouts can occur at any time. The number of burnouts in any time interval is independent of the number in any other time interval if and only if the two time intervals are non-overlapping.

   (a) If a light bulb burns out at any time during the day, I replace it immediately (thus, for example, there is a nonzero but very small probability of replacing 10,000 light bulbs in any given day). Unfortunately, I have only a three-pack of light bulbs; if more than three light bulbs burn out this week, I will have to go to the store to buy more. What is the probability that I will get through the week (7 days) without going to the store? Your answer should be a number.

   Solution: The total expected number of burnouts per day is 0.4. Non-overlapping time intervals are still independent, so this is just a Poisson process with $\lambda = 0.4$. The probability of getting 3 or fewer total burnouts in 7 days is therefore

$$
\Pr\{X_{7\lambda} \le 3\} = \sum_{k=0}^{3} \frac{(2.8)^k e^{-2.8}}{k!} = e^{-2.8}(1 + 2.8 + 3.92 + 3.66) = 0.692
$$

   (b) I'm tired of changing my own light bulbs, so I'm going to hire a light-bulb-changing service. If one or more light bulbs burn out on any given day, a Light Bulb Technologist (certified by the LBTAA) will visit my house at 8:30pm that evening to change all of the broken bulbs; if no bulbs burn out that day, then the technologist does not visit. I pay a monthly charge that covers up to one visit per week; if the technologist has to visit my house more than once in any given week, I pay an emergency surcharge. What's the probability that I can get through any given week without paying an emergency surcharge? Your answer should be a number.

Solution: The probability that no light bulbs burn out on any given day is $e^{-0.4} = 0.67$. The probability that one or more bulbs burn out on any given day, and that a technologist visits my house, is $(1 - 0.67) = 0.33$. The number of days on which a technologist visits my house is a binomial random variable Y with

$$p_Y(j) = \begin{cases} \binom{7}{j} (0.33)^j (0.67)^{7-j} & 0 \le j \le 7 \\ 0 & \text{otherwise} \end{cases}$$

The probability that $Y \le 1$ is

$$p_Y(0) + p_Y(1) = (0.67)^7 + 7(0.33)(0.67)^6 = 0.27$$

4. **[PDF Scaling]**

Suppose that X, Y, and Z are the sampled values of three different audio signals. The mean and variance of an audio signal are uninteresting: the mean tells you the bias voltage of the microphone, and the variance tells you the signal loudness. For this reason, most audio signals are pre-normalized so that

$$E[X] = E[Y] = E[Z] = 0, \tag{1}$$

and

$$\text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = 1 \tag{2}$$

Suppose that the signals X, Y, and Z have been normalized as shown in Eqs. 1 and 2, and now you are trying to find out which of these three signals is the most spiky, where spiky is defined as follows:

A Spiky Random Variable is a random variable that generates "very large" values at least once in every hundred trials, i.e., X is spiky if and only if $\Pr\{|X| > 3\sigma_X\} > 0.01$.

(a) Suppose that X is a zero-mean, unit-variance Gaussian random variable. (1) What is $\Pr\{|X| > \sigma_X\}$? (2) What is $\Pr\{|X| > 3\sigma_X\}$? (3) Is X spiky? Be sure to consider both positive and negative values of the random variable.

Solution: From the CDF tables on pages 237-8,

$$\Pr\{|X| > \sigma_X\} = 2\Pr\{X > \sigma_X\} = 2Q(1) = 0.3174$$

$$\Pr\{|X| > 3\sigma_X\} = 2\Pr\{X > 3\sigma_X\} = 2Q(3) = 0.0026$$

Therefore the Gaussian random variable is not spiky.

(b) Suppose that Y is a uniform random variable, scaled so that it has zero mean and unit variance. (1) What is $\Pr\{|Y| > \sigma_Y\}$? (2) What is $\Pr\{|Y| > 3\sigma_Y\}$? (3) Is Y spiky? Be sure to consider both positive and negative values of the random variable.

Solution: The mean and variance of a variable uniformly distributed between a and b are

$$E[Y] = \frac{a+b}{2}, \quad \sigma_Y^2 = \frac{(b-a)^2}{12}$$

Thus to set $E[Y] = 0$ and $\sigma_Y^2 = 1$, we need $b = -a = \sqrt{3}$. The pdf of this variable is therefore

$$f_Y(u) = \begin{cases} \frac{1}{2\sqrt{3}} & -\sqrt{3} \le u \le \sqrt{3} \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\Pr\{|Y| > \sigma_Y\} = 2\Pr\{Y > \sigma_Y\} = 2\int_1^{\sqrt{3}} \frac{1}{2\sqrt{3}} du = \frac{\sqrt{3}-1}{\sqrt{3}}$$

$$\Pr\{|Y| > 3\sigma_Y\} = 0$$

So the uniform random variable is definitely not spiky.

(c) A Laplacian random variable has the following pdf:

$$f_Z(u) = \frac{\lambda}{2} e^{-\lambda|u-\mu|}, \quad -\infty < u < \infty$$

Suppose that Z is a Laplacian random variable, with λ and μ chosen so that $E[Z] = 0$ and $\mathrm{Var}(Z) = 1$. (1) What is $\Pr\{Z > \sigma_Z\}$? (2) What is $\Pr\{Z > 3\sigma_Z\}$? (3) Is Z spiky? Be sure to consider both positive and negative values of the random variable.

Solution: A Laplacian is symmetric around μ, so to set $E[Z] = 0$, we choose $\mu = 0$. The variance of a Laplacian is then

$$\begin{aligned} E[Z^2] &= \int_{-\infty}^{\infty} \frac{\lambda u^2}{2} e^{-\lambda|u|} du \\ &= \int_0^{\infty} \lambda u^2 e^{-\lambda u} du, \end{aligned}$$

which is the same as the second moment of an exponential random variable, thus $E[Z^2] = \frac{2}{\lambda^2}$. In order to set $E[Z^2] = 1$, therefore, we should choose $\lambda = \sqrt{2}$.

The question in the problem set asks for $\Pr\{Z > 3\sigma_Z\}$, but in order to find out whether or not the RV is spiky, you need to find $\Pr\{|Z| > \sigma_Z\}$. We will accept either answer as correct:

$$\Pr\{|Z| > \sigma_Z\} = 0.243$$

OR

$$\Pr\{Z > \sigma_Z\} = 0.1215$$

$$\Pr\{|Z| > 3\sigma_Z\} = 0.014$$

OR

$$\Pr\{Z > 3\sigma_Z\} = 0.007$$

So the Laplacian random variable IS spiky!

5. **[Gaussian CDF and Complementary CDF]**
Professors often assign grades based on the assumption that student test scores are Gaussian-distributed. For example, a professor might want to give an A grade to 25 percent of the students, and in order to accomplish this goal, he or she might announce that A grades will be given to all students who have scores at least 0.68 standard deviations above the mean. The problem with this logic is that a Gaussian random variable can be arbitrarily large, whereas

student test scores never exceed 100 points (on a 100-point test). Suppose, for example, that Professor Schmoe gives a 100-point test, and afterward determines that the test score, X, is a random variable with $E[X] = 70$ and $\sigma_X = 18$.

(a) Suppose that Y is a Gaussian random variable with $E[Y] = E[X]$ and $\sigma_Y = \sigma_X$. What is $\Pr\{Y > 100\}$? Round your answer off to the nearest entry in either Table 6.1 or Table 6.2.

Solution:

$$\Pr\{Y > 100\} = Q\left(\frac{100 - E[Y]}{\sigma_Y}\right) \approx Q(1.67) = 0.0475$$

(b) Professor Schmoe argues that assigning grades using a Gaussian curve is justified, because the missing upper tail of the Gaussian (the scores above 100 points that would exist if X were Gaussian, but that do not exist in reality) is balanced by the missing lower tail (the scores below 0 points that would exist if X were Gaussian). You argue, correctly, that Prof. Schmoe's argument is completely unjustified, because $\Pr\{Y \leq 0\}$ is much, much less than $\Pr\{Y > 100\}$. In order to make this argument, you'd better figure out what $\Pr\{Y \leq 0\}$ is. Round your answer off to the nearest entry in either Table 6.1 or Table 6.2. **Solution:**

$$\Pr\{Y \leq 0\} = \Phi\left(\frac{0 - E[Y]}{\sigma_Y}\right) \approx \Phi(-3.8) = Q(3.8) = 0.0000724$$