ECE 313: Problem Set 7: Problems and Solutions
CDF and pdf; Uniform and Exponential random variables

Due: Wednesday, March 6 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.1–3.4

1. **[Cumulative Distribution Function]**
For each of the following functions $F_i(c)$, state whether or not $F_i(c)$ is the CDF of some random variable. If not, state which of the properties of a CDF it violates. If so, find the corresponding pmf or pdf.

(a)

$$F_1(c) = \begin{cases} 0 & c \leq 0 \\ 0.5c & 0 < c \leq 1 \\ 0.25 + 0.25c & 1 < c \leq 3 \\ 1 & 3 < c \end{cases}$$

Solution: This is a valid CDF; the corresponding pdf is

$$f_1(u) = \begin{cases} 0.5 & 0 < c < 1 \\ 0.25 & 1 < c < 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$F_2(c) = \begin{cases} 0 & c \leq 0 \\ 0.5 & 0 < c \leq 1 \\ 0.75 & 1 < c \leq 3 \\ 1 & 3 < c \end{cases}$$

Solution: This is not a valid CDF, because it is not right-continuous.

(c)

$$F_3(c) = \begin{cases} 0.5 & c < 1 \\ 0.75 & 1 \leq c < 3 \\ 1 & 3 \leq c \end{cases}$$

Solution: This is not a valid CDF, because it does not approach 0.0 as $c \to -\infty$.

(d)

$$F_4(c) = \begin{cases} 0 & c < 0 \\ 0.5 & 0 \leq c < 1 \\ 0.75 & 1 \leq c < 3 \\ 1 & 3 \leq c \end{cases}$$

Solution: This is a valid CDF. The corresponding pmf is

$$p_4(k) = \begin{cases} 0.5 & k = 0 \\ 0.25 & k = 1 \\ 0.25 & k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$F_5(c) = \begin{cases} 0 & c < 0 \\ 0.5 & 0 \le c < 1 \\ 0.25 & 1 \le c < 3 \\ 1 & 3 < c \end{cases}$$

Solution: This is not a valid CDF, because it is not non-decreasing.

(f)

$$F_6(c) = \begin{cases} 0 & c \le 0 \\ 0.5c & 0 < c \le 1 \\ 0.25 + 0.25c & 1 < c \end{cases}$$

Solution: This is not a valid CDF, because it does not approach 1.0 as $c \to \infty$.

2. **[Continuous Random Variables]**
Random variable X is distributed with the following pdf:

$$f_X(u) = \begin{cases} \sin(x) & 0 \le x \le A \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of the constant A? **Solution:** $f_X(u)$ must integrate to one. We have that

$$\int_{-\infty}^{\infty} f_X(u)du = \int_0^A \sin(x) = 1 - \cos(A)$$

Thus, to achieve $\int_{-\infty}^{\infty} f_X(u)du = 1$, we need $A = \frac{\pi}{2}$.

(b) What is the corresponding CDF, $F_X(c)$? **Solution:**

$$F_X(c) = \int_{-\infty}^{c} f_X(u)du = \begin{cases} 0 & c \le 0 \\ 1 - \cos(c) & 0 \le c \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \le c \end{cases}$$

(c) What is $E[X]$? **Solution:**

$$E[X] = \int_0^{\pi/2} u\sin(u)du = [-u\cos(u)]_0^{\pi/2} + \int_0^{\pi/2} \cos(u)du = 1$$

(d) What is $\text{Var}(X)$? **Solution:**

$$\begin{aligned} E[X^2] &= \int_0^{\pi/2} u^2 \sin(u)du \\ &= \left[-u^2\cos(u)\right]_0^{\pi/2} + 2\int_0^{\pi/2} u\cos(u)du \\ &= 0 + 2\left[u\sin(u)\right]_0^{\pi/2} - 2\int_0^{\pi/2} \sin(u)du \\ &= \pi - 2 \end{aligned}$$

Therefore

$$\text{Var}(X) = E[X^2] - E^2[X] = \pi - 3$$

3. **[Independent intervals and the exponential pdf]**
Consider a process in which successes can occur at any time. The number of successes occurring in any time interval is independent of the number of successes occurring in any other time interval unless the intervals overlap. Define

$$G_1(u) \equiv \Pr\{T_1 > u\}, \quad L_1(u) \equiv \ln \Pr\{T_1 > u\}$$

Consider the two non-overlapping intervals $(0, u]$ and $(u, v]$. Since these two intervals are non-overlapping, what happens in the second interval is independent of what happens in the first interval, as long as $u \geq 0$ and $v \geq u$. Let A be the event that there are no successes in the first interval, and let B be the event that there are no successes in the second interval. Since A and B are independent events,

$$P(AB) = P(A)P(B)$$

But notice that $P(A) = G_1(u)$, $P(B) = G_1(v-u)$, and $P(AB) = G_1(v)$, therefore

$$G_1(v) = G_1(u)G_1(v-u), \quad u \geq 0, \quad v \geq u \tag{1}$$

or equivalently,

$$L_1(v) = L_1(u) + L_1(v-u), \quad u \geq 0, \quad v \geq u$$

(a) According to Eq. 1, what must be the dependence of $L_1(u)$ on u, for $u > 0$? Your answer should be one word.
Solution: Linear.

(b) Consider the following equations:

$$\begin{aligned} P\{T_1 > 0\} &= 1 \quad (2) \\ P\{T_1 > \infty\} &= 0 \quad (3) \end{aligned}$$

Describe the set of all functions $G_1(u)$ that satisfy Eq. 1, Eq. 2, and Eq. 3.
Solution: The only functions that satisfy these three conditions are functions of the form $G_1(u) = e^{-\lambda u}$ for positive u, and for some real-valued constant λ.

(c) What is the relationship between the CDF, $F_T(u)$, and the function $G_1(u)$?
Solution: $F_T(u) = 1 - G_1(u)$.

4. **[Central moments of the uniform]**
A uniform random variable is defined by the pdf

$$f_X(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{otherwise} \end{cases}$$

For the purpose of calculating its central moments, though, it is more useful to define it as

$$f_X(u) = \begin{cases} \frac{1}{\Delta} & \mu - \frac{\Delta}{2} \leq u \leq \mu + \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

where the relationship between the parameters (a, b) and the parameters (μ, Δ) is given by

$$\mu = \frac{a+b}{2}, \quad \Delta = b - a$$

(a) In terms of Δ, find the n^{th} central moment, $E[(X-\mu)^n]$, for odd integer values of n.

Solution:

$$\begin{aligned}
E[(X-\mu)^n] &= \int_{\mu-\Delta/2}^{\mu+\Delta/2} (u-\mu)^n f_X(u)du \\
&= \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} v^n dv \\
&= \frac{1}{(n+1)\Delta}\left[v^{n+1}\right]_{-\Delta/2}^{\Delta/2} \\
&= \frac{1}{(n+1)\Delta}\left(\left(\frac{\Delta}{2}\right)^{n+1} - \left(-\frac{\Delta}{2}\right)^{n+1}\right)
\end{aligned}$$

For odd-valued n, the last line simplifies to $E[(X-\mu)^n] = 0$.

(b) In terms of Δ, find the n^{th} central moment, $E[(X-\mu)^n]$, for even integer values of n.

Solution:

$$E[(X-\mu)^n] = \frac{1}{n+1}\left(\frac{\Delta}{2}\right)^n$$

(c) Express the standard deviation σ_X in terms of Δ.

Solution: $\sigma_X = \sqrt{E[(X-\mu)^2]} = \frac{1}{\sqrt{3}}\frac{\Delta}{2}$