University of Illinois **Spring 2013**

ECE 313: Problem Set 5: Solutions
Confidence Interval, Bayes formula, Hypothesis testing

1. **[Confidence interval]**

   (a) The error count E can be obtained by running n independent trials of a Bernoulli random variable with parameter p_e. Thus, E is a binomial random variable with parameters (n, p_e).

   (b) The probability of error p_e should lie in the range $\hat{p}_e - 10^{-5} \leq p_e \leq \hat{p}_e + 10^{-5}$ in order to meet a $\pm 10\%$ tolerance. For a 99% confidence level, i.e., $a = 10$,

   $$\frac{a}{2\sqrt{n}} = 10^{-5}$$

   Thus, $n = 25 \times 10^{10}$ bits are needed.

   (c) The desk-top CPU executes 4×10^9 cycles/sec. Thus, $\frac{4\times 10^9}{40} = 10^8$ bits are simulated per sec. As there are $n = 25 \times 10^{10}$ bits, one simulation run will take $\frac{25\times 10^{10}}{60\times 10^8} = 41.6$ minutes.

   (d) The simulation run takes 600 seconds, which corresponds to 24×10^{11} clock cycles. Thus, $n = \frac{24\times 10^{11}}{40} = 6 \times 10^{10}$ bits can be simulated, leading to

   $$\frac{a}{2\sqrt{6 \times 10^{10}}} = 10^{-5} \Rightarrow a = 2\sqrt{6}$$

   and thus a confidence level of

   $$1 - \frac{1}{a^2} = 1 - \frac{1}{24} = 0.958$$

   i.e., a 95.8% confidence level. The designer considers this loss in confidence in the initial runs as a reasonable compromise in order to get the project completed on time.

2. **[Bayes Formula]**

   (a) $P(T) = P(T \mid \text{fair coin})P(\text{fair coin}) + P(T \mid \text{biased coin})P(\text{biased coin})$
   $= \frac{1}{2} \cdot \frac{2}{3} + (1-p) \cdot \frac{1}{3} = \frac{7}{12} \Rightarrow p = \frac{1}{4}$.

   (b) $P(\text{one Head, one Tail} \mid \text{two fair coins}) = P(\{HT, TH\}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$.
   $P(\text{one Head, one Tail} \mid \text{one fair, one biased})$
   $= P(\{\text{fair} = H, \text{biased} = T\}) + P(\{\text{biased} = H, \text{fair} = T\}) = \frac{1}{2} \cdot (1-p) + p \cdot \frac{1}{2} = \frac{1}{2}$ regardless of the value of p. Therefore, the theorem of total probability gives

   $$\begin{aligned} P(\text{one Head, one Tail}) &= P(H, T \mid \text{two fair coins})P(\text{two fair coins}) \\ &\quad + P(H, T \mid \text{one fair, one biased})P(\text{one fair, one biased}) \\ &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}, \end{aligned}$$

   and the *conditional* probability of both coins being fair given that one Head and one Tail was observed is, by Bayes' formula,

   $$P(\text{both coins fair}|\text{one Head, one Tail}) = \frac{P(H, T|\text{two fair coins})P(\text{two fair coins})}{P(\text{one Head, one Tail})} = \frac{1/6}{1/2} = \frac{1}{3}$$

   the same as the unconditional probability! What does this tell you about the events {both coins fair} and {one Head, one Tail}?

3. **[Law of Total Probability]**

(a) Tom goes home after 3 tosses with \$8 if the outcomes of the tosses is HHH. The probability of this event is $\frac{1}{8}$.

(b) Tom goes home after 5 tosses with \$8 if the outcomes of the tosses is any of THHHH, HTHHH, HHTHH. Tom goes home after 5 tosses with \$ 0 if the outcome of the tosses is TTTTT. So the probability that Tom goes home after 5 tosses is

$$P\{\text{TTTTT, THHHH, HTHHH, HHTHH}\} = 4\cdot\left(\frac{1}{2}\right)^5 = \frac{1}{8}.$$

.

(c) Since Tom tosses the coin at least 5 times, we know that he did not go home after 3 tosses with \$ 8 (which event has probability 1/8 as calculated in part (a)). It is straightforward to determine that Tom does not go home after 1, 2 or 4 coin tosses. So, the probability of the conditioning event A — Tom tossed the coin at least 5 times — is $P(A) = 7/8 = 28/32$.

We have already calculated that $P(\{\mathbb{X}=0\}\cap A) = \frac{1}{32}$ and $P(\{\mathbb{X}=8\}\cap A) = \frac{3}{32}$ in part (b). The other possibilities are that Tom's *wealth* is

- \$2 (1 Head and 4 Tails) which has probability $P(\{\mathbb{X}=2\}\cap A) = \binom{5}{1}\cdot\left(\frac{1}{2}\right)^5 = 5\cdot\left(\frac{1}{2}\right)^5 = \frac{5}{32}$;
- \$4 (2 Heads and 3 Tails) which has probability $P(\{\mathbb{X}=4\}\cap A) = \binom{5}{2}\cdot\left(\frac{1}{2}\right)^5 = 10\cdot\left(\frac{1}{2}\right)^5 = \frac{10}{32}$;
- \$6 (3 Heads and 2 Tails in 5 tosses *except* for outcome HHHTT which cannot occur) which has probability $P(\{\mathbb{X}=6\}\cap A) = \left(\binom{5}{3}-1\right)\cdot\left(\frac{1}{2}\right)^5 = 9\cdot\left(\frac{1}{2}\right)^5 = \frac{9}{32}$.

Since $p_{\mathbb{X}|A}(i \mid A) = \dfrac{P(\{\mathbb{X}=i\}\cap A)}{P(A)} = P(\{\mathbb{X}=i\}\cap A) \times \dfrac{32}{28}$, we get that

$$p_{\mathbb{X}|A}(0 \mid A) = \frac{1}{28};\quad p_{\mathbb{X}|A}(2 \mid A) = \frac{5}{28};\quad p_{\mathbb{X}|A}(4 \mid A) = \frac{10}{28};\quad p_{\mathbb{X}|A}(6 \mid A) = \frac{9}{28};\quad p_{\mathbb{X}|A}(8 \mid A) = \frac{3}{28}.$$

4. **[Bayes Formula]**

(a) Let A denote the event that Harry accepts the candidate. Then, we are given that $P(H) = \frac{1}{3}$; $P(N) = \frac{2}{3}$ and that $P(A \mid H) = \frac{3}{4}$ and $P(A \mid N) = \frac{3}{40}$. Hence, by the law of total probability,

$$P(A) = P(A \mid H)P(H) + P(A \mid N)P(N) = \frac{3}{4}\times\frac{1}{3} + \frac{3}{40}\times\frac{2}{3} = \frac{1}{4} + \frac{1}{20} = \frac{5+1}{20} = \frac{6}{20} = \frac{3}{10}.$$

(b) Using Bayes' formula, $P(H \mid A) = \dfrac{P(A \mid H)P(H)}{P(A)} = \dfrac{5/20}{6/20} = \dfrac{5}{6}$.

Note that all the numbers used are from the law of total probability calculation done above.

5. **[The (in)famous Monty Hall problem]**

(a) When you first pick, all doors are equally likely, so $P(A) = \frac{1}{3}$.

(b) For the stay-put strategy, $P(B \mid A) = 1$, and $P(B \mid A^c) = 0$. Using Equation 3.4 in the book, a.k.a. the law of total probability, $P(B) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c) = \frac{1}{3}$.

(c) For the always-switch strategy, $P(B \mid A) = 0$, $P(B \mid A^c) = 1$, and so we get $P(B) = \frac{2}{3}$.

Note that Monty's offer is essentially "Would you rather have what is behind the two doors that you didn't pick (except you don't have to take the goat that has already been revealed home with you), or would you rather stay with your original pick where you have a 1/3 chance of having picked the right door?" Clearly, the always-switch strategy is best.

(d) If we pick with equal probabilities $\frac{1}{2}$ one of the two unopened doors, we have that $P(B \mid A) = P(B \mid A^c) = \frac{1}{2}$, and so $P(B) = \frac{1}{2}$.

6. **[Hypothesis Testing]**

(a) $P(T+) = P(T+|H+)P(H+) + P(T+|H-)P(H-)$.
Test A: $P(T+) = (0.999)(0.02) + (0.01)(0.98) = 0.02978$.
Test B: $P(T+) = (0.99)(0.02) + (0.001)(0.98) = 0.02078$.

(b) $P(H+|T+) = \frac{P(H+,T+)}{P(T+)} = \frac{P(H+)P(T+|H+)}{P(T+)}$.
Test A: $P(H+|T+) = \frac{(0.02)(0.999)}{(0.02978)} = 0.67092$.
Test B: $P(H+|T+) = \frac{(0.02)(0.99)}{(0.02078)} = 0.952839$.

(c) $P(H+|T-) = \frac{P(H+)P(T-|H+)}{P(T-)} = \frac{P(H+)[1-P(T+|H+)]}{[1-P(T+)]}$.
Test A: $P(H+|T-) = \frac{(0.98)(1-0.999)}{(1-0.2978)} = 0.00101$.
Test B: $P(H+|T-) = \frac{(0.98)(1-0.99)}{(1-0.2078)} = 0.010008$.

7. **[Hypothesis testing]**

(a) The likelihood matrix L is as shown below and the maximum-likelihood decision rule is indicated shading.

Hypothesis	$\mathbb{X}=0$	$\mathbb{X}=1$	$\mathbb{X}=2$	$\mathbb{X}=3$
H_1	0.4	0.3	0.2	0.1
H_0	0.1	0.2	0.3	0.4

It is easy to get $P_{\text{FA}} =$ sum of unshaded entries on H_0 row $= 0.1 + 0.2 = 0.3$
and $P_{\text{MD}} =$ sum of unshaded entries on H_1 row $= 0.1 + 0.2 = 0.3$ also.

(b) By the law of total probability, $P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = 0.7 \times 0.3 + 0.3 \times 0.3 = 0.3$.

(c) The joint probability matrix $J = LD$ where $D =$ diag$[\pi_1, \pi_0]$ is as shown below, and the Bayesian decision rule is as indicated by the shading.

Hypothesis	$\mathbb{X}=0$	$\mathbb{X}=1$	$\mathbb{X}=2$	$\mathbb{X}=3$
H_1	0.12	0.09	0.06	0.03
H_0	0.07	0.14	0.21	0.28

$P(E) =$ sum of all the unshaded entries in the J matrix $= 0.25$ is smaller than the average error probability for the ML rule, as it should be.

(d) In general, the J matrix can be written as

Hypothesis	$\mathbb{X}=0$	$\mathbb{X}=1$	$\mathbb{X}=2$	$\mathbb{X}=3$
H_1	$0.4\pi_1$	$0.3\pi_1$	$0.2\pi_1$	$0.1\pi_1$
H_0	$0.1\pi_0$	$0.2\pi_0$	$0.3\pi_0$	$0.4\pi_0$

Now, if $0.1\pi_0 > 0.4\pi_1 = 0.4(1-\pi_0)$, that is, $\pi_0 > 0.8$, then the Bayesian decision is in favor of H_0 whenever $\mathbb{X} = 0$. But, if $\pi_0 > 0.8$, then the same decision (favoring H_0) holds whenever $\mathbb{X}$ equals 1, 2, or 3 also. (Work it out!) Hence, if $\pi_0 > 0.8$, the Bayesian decision always is in favor of H_0. From the symmetry of the problem, it should be obvious that if $\pi_1 > 0.8$, the Bayesian decision always is in favor of H_1. The incredulous are invited to work out the details for themselves.

Alternatively, the *likelihood ratio* takes on values, 4, $\frac{3}{2}$, $\frac{2}{3}$ and $\frac{1}{4}$, and hence always exceeds $\frac{\pi_0}{\pi_1} = \frac{\pi_0}{1-\pi_0}$ if $\pi_0 < 0.2$, that is, if $\pi_1 > 0.8$, and can never exceed $\frac{\pi_0}{\pi_1} = \frac{\pi_0}{1-\pi_0}$ if $\pi_0 > 0.8$. The result to be proved follows from this.