ECE 313: Problem Set 4: Problems and Solutions
Binomial, Geometric, Poisson and Negative Binomial Distributions; Bernoulli Process

Due: Wednesday, February 13 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.5–2.7

1. **[Bernoulli process]**
Consider a process in which there is one trial per click (a click is a unit of time), starting with click number 1. Each trial results in success with probability p, or in failure with probability $1-p$. Define the following random variables:

   - X_m is the number of successes achieved in the first m clicks, i.e., during the time interval $t \in \{1, \ldots, m\}$.
   - X_n is the number of successes achieved in the first n clicks, i.e., during the time interval $t \in \{1, \ldots, n\}$, where $n > m$ in all parts of this problem.
   - T_j is the time (click index) at which the j^{th} success occurs.
   - T_k is the time (click index) at which the k^{th} success occurs, where $k > j$ in all parts of this problem.

   In terms of the variables p, n, j, k, etc., find the following pmfs and conditional pmfs:

   (a) $p_{X_n}(k)$ is the probability that there are k successes in the first n clicks. Write a formula for $p_{X_n}(k)$.
   Solution: X_n is a binomial random variable, with pmf

$$p_{X_n}(k) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

   (b) $p_{X_n|X_m}(k|i)$ is the probability that there are k successes in the first n clicks, given that there are i successes in the first m clicks, where $m < n$. Write a formula for $p_{X_n|X_m}(k|i)$.
   Solution: There are i successes in the interval $t \in \{1, \ldots, m\}$, therefore there must be $k-i$ successs in the interval $t \in \{m+1, \ldots, n\}$. Non-overlapping intervals of time are independent, therefore this is just another binomial random variable, with parameters $n-m$ and p:

$$p_{X_n|X_m}(k|i) = \begin{cases} \begin{pmatrix} n-m \\ k-i \end{pmatrix} p^{k-i} (1-p)^{(n-m)-(k-i)} & 0 \le k-i \le n-m \\ 0 & \text{otherwise} \end{cases}$$

   (c) $p_{X_m|X_n}(i|k)$ is the probability that there are i successes in the first m clicks, given that there are k successes in the first n clicks, where $m < n$. Write a formula for $p_{X_m|X_n}(i|k)$.
   Solution: The interval $t \in \{1, \ldots, m\}$ is a strict subset of the interval $t \in \{1, \ldots, n\}$, so the non-overlapping intervals argument used in part (b) does not work here. Indeed,

there are $\begin{pmatrix} n \\ k \end{pmatrix}$ ways that we could arrange k successes into n clicks, and each of those patterns is equally likely (each such sequence has *a priori* probability $p^k(1-p)^{n-k}$). Of those patterns, there are $\begin{pmatrix} m \\ i \end{pmatrix}$ ways that we could arrange i successes in m clicks, and there are $\begin{pmatrix} n-m \\ k-i \end{pmatrix}$ ways that we could arrange $k-i$ successes in $n-m$ clicks, therefore

$$p_{X_m|X_n}(i|k) = \begin{cases} \dfrac{\begin{pmatrix} m \\ i \end{pmatrix}\begin{pmatrix} n-m \\ k-i \end{pmatrix}}{\begin{pmatrix} n \\ k \end{pmatrix}} & 0 \le i \le k \\ 0 & \text{otherwise} \end{cases}$$

(d) $p_{T_1}(u)$ is the probability that the first success happens on the u^{th} click. Write a formula for $p_{T_1}(u)$.
Solution: This is a geometric random variable, with pmf

$$p_{T_1}(u) = \begin{cases} p(1-p)^{u-1} & u \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

(e) $p_{T_k}(v)$ is the probability that the k^{th} success happens on the v^{th} click. Write a formula for $p_{T_k}(v)$.
Solution: This is a negative binomial random variable, with pmf

$$p_{T_1}(v) = \begin{cases} \begin{pmatrix} v-1 \\ k-1 \end{pmatrix} p^k(1-p)^{v-k} & v \ge k \\ 0 & \text{otherwise} \end{cases}$$

(f) $p_{T_k|T_j}(v|u)$ is the probability that the k^{th} success happens on the v^{th} click, given that the j^{th} success happened on the u^{th} click, where $j<k$. Write a formula for $p_{T_k|T_j}(v|u)$.
Solution: The time intervals $\{1,\ldots,u\}$ and $\{u+1,\ldots,v\}$ are non-overlapping, therefore the events that occur during these two time intervals are independent. The probability $p_{T_k|T_j}(v|u)$ is therefore identical to the probability of getting the $(k-j)^{\text{th}}$ success on the $(v-u)^{\text{th}}$ click, i.e.,

$$p_{T_k|T_j}(v|u) = \begin{cases} \begin{pmatrix} v-u-1 \\ k-j-1 \end{pmatrix} p^{k-j}(1-p)^{(v-u)-(k-j)} & v-u \ge k-j \\ 0 & \text{otherwise} \end{cases}$$

(g) $p_{T_j|T_k}(u|v)$ is the probability that the j^{th} success happens on the u^{th} click, given that the k^{th} success happened on the v^{th} click, where $j<k$. Write a formula for $p_{T_j|T_k}(u|v)$.
Solution: The time interval $\{1,\ldots,u\}$ is a strict subset of the interval $\{1,\ldots,v\}$, therefore the independence argument used in part (f) won't work here. Indeed, there are $\begin{pmatrix} v-1 \\ k-1 \end{pmatrix}$ different sequences that have their k^{th} success on the v^{th} click, and all of

these particular sequences are equally likely (the *a priori* probability of each particular sequence matching this criterion is $p^k(1-p)^{v-k}$). Of these particular sequences, there are $\binom{u-1}{j-1}$ different ways of arranging the first $(j-1)$ successes in the first $(u-1)$ time steps, and there are $\binom{v-u-1}{k-j-1}$ different ways of arranging the remaining $(k-j-1)$ successes on the remaining $(v-u-1)$ clicks, thus

$$p_{T_j|T_k}(u|v) = \begin{cases} \dfrac{\binom{u-1}{j-1}\binom{v-u-1}{k-j-1}}{\binom{v-1}{k-1}} & j \le u \le v-(k-j) \\ 0 & \text{otherwise} \end{cases}$$

2. **[Serial and parallel Bernoulli processes]**
I have a very small house; there are only 4 light bulbs in my house. On any given day, each of these light bulbs has a probability $p = 0.1$ of burning out, and a probability $(1-p) = 0.9$ of continuing to function.

(a) If a light bulb burns out at any time during the day, I replace it at 8:30pm that evening. Each light bulb is replaced at most once per day. Unfortunately, I have only a three-pack of light bulbs; if more than three light bulbs burn out this week, I will have to go to the store to buy more. What is the probability that I will get through the week (7 days) without going to the store? Your answer should be an equation in terms of constants, e.g., it may include terms like $(0.063)^{45}$ and $\binom{320}{6}$, but it may not include terms like p^{45} and $\binom{n}{6}$.

Solution: This is a Bernoulli process with $7 \times 4 = 28$ trials (up to 28 light bulbs might burn out), each of which has a failure probability of $p = 0.1$. The probability that three or fewer bulbs will burn out this week is therefore

$$P\{X_{28,0.1} \le 3\} = (0.9)^{28} + 28(0.1)(0.9)^{27} + \binom{28}{2}(0.1)^2(0.9)^{26} + \binom{28}{3}(0.1)^3(0.9)^{25}$$

(b) I'm tired of changing my own light bulbs, so I'm going to hire a light-bulb-changing service. If one or more light bulbs burn out on any given day, a Light Bulb Technologist (certified by the LBTAA) will visit my house at 8:30pm that evening to change all of the broken bulbs; if no bulbs burn out that day, then the technologist does not visit. I pay a monthly charge that covers up to one visit per week; if the technologist has to visit my house more than once in any given week, I pay an emergency surcharge. What's the probability that I can get through any given week without paying an emergency surcharge? Your answer may include terms like $(0.063)^{45}$ and $\binom{320}{6}$, but it may not include terms like p^{45} and $\binom{n}{6}$.

Solution: The probability that no light bulbs burn out on any given day is $(0.9)^4$. The probability that one or more bulbs burn out on any given day, and that a technologist visits my house, is $(1-(0.9)^4)$. The number of days on which a technologist visits my house is a binomial random variable Y with

$$p_Y(j) = \begin{cases} \binom{7}{j} (1-(0.9)^4)^j (0.9)^{4(7-j)} & 0 \le j \le 7 \\ 0 & \text{otherwise} \end{cases}$$

The probability that $Y \le 1$ is

$$p_Y(0) + p_Y(1) = (0.9)^{28} + 7(0.9)^{24}(1-(0.9)^4)$$

3. **[Moments of the binomial and Poisson]**
In terms of its parameters n and p, the binomial pmf is given by

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

In terms of its parameter λ, the Poisson pmf is given by

$$p_Y(j) = \begin{cases} \frac{\lambda^j e^{-\lambda}}{j!} & 0 \le j < \infty \\ 0 & \text{otherwise} \end{cases}$$

It is possible to compute the third moment of a binomial or Poisson pmf by writing the expected-value formula,

$$E[X^3] = \sum_{k=-\infty}^{\infty} k^3 p_X(k),$$

and then reducing the summation to a few terms that look like

$$1 = \sum_{i=0}^{m} \binom{m}{i} p^i (1-p)^{m-i},$$

or

$$1 = \sum_{i=0}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!}$$

using methods similar to those demonstrated in Eq. (2.7) in the textbook. HINT: The following formula is also useful:

$$k^3 = k(k-1)(k-2) + 3k(k-1) + k$$

(a) Find the third moment of a binomial pmf, $E[X^3]$, in terms of the parameters n and p. Show your work; correct answers given with no work will receive zero points.

Solution:

$$
\begin{aligned}
E[X^3] &= \sum_{k=0}^{n} k^3 p_X(k) \\
&= \sum_{k=0}^{n} \left(k(k-1)(k-2) + 3k(k-1) + k\right) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=3}^{n} \frac{n!}{(k-3)!(n-k)!} p^k (1-p)^{n-k} + 3 \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} \\
&+ \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1)(n-2)p^3 \sum_{k=3}^{n} \begin{pmatrix} n-3 \\ k-3 \end{pmatrix} p^{k-3} (1-p)^{n-k} \\
&+ 3n(n-1)p^2 \sum_{k=2}^{n} \begin{pmatrix} n-2 \\ k-2 \end{pmatrix} p^{k-2} (1-p)^{n-k} + np \sum_{k=1}^{n} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} p^{k-1} (1-p)^{n-k} \\
&= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np
\end{aligned}
$$

(b) Find the third moment of a Poisson pmf, $E[Y^3]$, in terms of the parameter λ. Show your work; correct answers given with no work will receive zero points.
Solution:

$$
\begin{aligned}
E[Y^3] &= \sum_{i=0}^{\infty} i^3 p_Y(i) \\
&= \sum_{i=0}^{\infty} \left(i(i-1)(i-2) + 3i(i-1) + i\right) \frac{\lambda^i e^{-\lambda}}{i!} \\
&= \sum_{k=3}^{\infty} \frac{\lambda^i e^{-\lambda}}{(i-3)!} + 3 \sum_{i=2}^{\infty} \frac{\lambda^i e^{-\lambda}}{(i-2)!} + \sum_{i=1}^{\infty} \frac{\lambda^i e^{-\lambda}}{(i-1)!} \\
&= \lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{i-3} e^{-\lambda}}{(i-3)!} + 3\lambda^2 \sum_{i=2}^{\infty} \frac{\lambda^{i-2} e^{-\lambda}}{(i-2)!} + \lambda \sum_{i=1}^{\infty} \frac{\lambda^{i-1} e^{-\lambda}}{(i-1)!} \\
&= \lambda^3 + 3\lambda^2 + \lambda
\end{aligned}
$$

(c) On p. 45 of the lecture notes, it is suggested that the Poisson pmf with $\lambda = np$ is a good approximation for the binomial pmf. As you have seen in parts (a) and (b) of this problem, even if you set $\lambda = np$, $E[X^3] \neq E[Y^3]$. Which one is larger? Why does your answer make sense (HINT: what is the maximum possible value of X? What is the maximum possible value of Y?)

Solution: $E[Y^3] > E[X^3]$. This makes sense, because Y can take any non-negative integer value, while X is bounded to take only values less than or equal to n.

4. **[Random dating]**
Nikki has signed up for an on-line dating site. On this site, each member has a profile, and

members can send each other *cards*; when two members send cards back and forth to one another, the cards accumulate into a *conversation*. Nikki is a bit random; each of her cards has a 90% chance of being ordinary and unremarkable, but each card also has a 10% chance of being witty&wonderful (each card's content is chosen independently of the content of any other card, regardless of the length of the conversation). Any other member of the dating site who receives a witty&wonderful card from Nikki immediately agrees to meet her in person, and they stop exchanging cards (we do not know what happens to the relationship after that). Nikki is persistent; she will continue sending cards to a person until the person agrees to meet her.

(a) What is the average number of cards that Nikki sends to any individual before the conversation ends? What is the standard deviation?

Solution: This is a geometric random variable with parameter $p = 0.1$. $E[C] = 1/p = 10$, $\sigma_C = \sqrt{(1-p)/p^2} = \sqrt{90} \approx 9.5$.

(b) Nikki might send many ordinary cards, possibly to many different people, before she sends her first witty&wonderful card to anybody. What is the expected number of cards that Nikki will send, up to and including the first witty&wonderful card? What is the standard deviation of this quantity?

Solution: The probability of a witty&wonderful card is still $p = 0.1$, therefore this is also a geometric random variable with expected value of 10 and standard deviation of 3.

(c) Out of every 4 cards Nikki sends, let X be the number of cards that are witty&wonderful. Find the numerical values of all nonzero terms in the probability mass function $p_X(k)$.

Solution: X is a binomial random variable with parameters $p = 0.1$, $n = 4$. The nonzero terms in the pmf are $p_X(0) = 0.9^4 = 0.6561$, $p_X(1) = 4(0.1)(0.9)^3 = 0.2916$, $p_X(2) = 6(0.1)^2(0.9)^2 = 0.0486$, $p_X(3) = 4(0.1)^3(0.9) = 0.0036$, and $p_X(4) = (0.1)^4 = 0.0001$.

(d) So far, Nikki has found only three people to whom she would like to send cards. Let Y be the total number of cards Nikki will send before all three people agree to meet her. Write an equation expressing the form of the probability mass function $p_Y(v)$. Your expression should have no variables on the right-hand side except v.

Solution: This is a negative binomial random variable with parameters $p = 0.1$ and $k = 3$. Its pmf is

$$p_Y(v) = \binom{v-1}{2} (0.1)^3 (0.9)^{v-3}$$

(e) This part of the problem takes place two days later than part (d), and Nikki has come over to your apartment to brag about her magnetic on-line personality. She tells you that she has now met all three of the people to whom she was sending cards, after sending a total of only thirteen cards. Given that the *third* meeting was arranged after the thirteenth card, what is the conditional probability that Nikki sent three or fewer cards before arranging her *first* meeting?

Solution: The third witty&wonderful card was card #13, so 2 of the first 12 cards were also witty&wonderful. There are $\binom{12}{2} = 66$ ways in which those two cards could have been ordered relative to the ten ordinary cards. Each of these particular orderings has equal probability. Of those 66 orderings, there are 11 ways in which the first card could have been witty&wonderful, 10 in which the second card but not the first is witty&wondeful, and 9 in which the third card but not the first two is witty&wonderful, thus the probability that the first meeting occurs by the time she sends three cards is $(9+10+11)/66 = 5/11$.

(f) Nikki has decided to broaden her horizons by writing to many different people. In particular, today is a busy day for Nikki; she will send 100 cards. She would like to know what is the probability that 12 people will agree to meet her. If your calculator is like mine, it will have some trouble computing the quantity $\binom{100}{12}$. Fortunately, the Poisson pmf has a form that can be computed pretty easily in this case. Use the Poisson approximation of the binomial pmf in order to estimate the probability that exactly 12 people agree to meet with Nikki as a result of the 100 cards she sends today.

Solution: For $n = 100$, $p = 0.1$, and therefore $\lambda = np = 10$, the Poisson approximation to the binomial pmf is

$$p_X(12) \approx \frac{(10)^{12}e^{-10}}{12!} = \frac{4.5 \times 10^7}{4.79 \times 10^8} \approx 0.094$$