ECE 313: Problem Set 3: Problems and Solutions
LOTUS, Conditional probability, and Independence

Due: Wednesday, February 8 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.2–2.4.3

1. **[Random phases in a communication system]**
   A particular communication system transmits a perfectly sinusoidal carrier wave, $X = \sin(T)$, where T is measured in microseconds.

   A particular receiver samples the signal at a random time T, and measures just one output voltage, $X = \sin(T)$. T has the following pmf:

$$p_T(u) = \begin{cases} 0.2 & u \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}\} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find $E[T]$ and σ_T^2.
   Solution: $E[T] = \frac{5\pi}{2}$, $\sigma_T^2 = 2\pi^2$

   (b) Find $E[X]$, $E[X^2]$, and σ_X^2 without finding the pmf of X.
   Solution: Using LOTUS, we find that $E[X] = \frac{1}{5}(1-1+1-1+1) = \frac{1}{5}$, $E[X^2] = \frac{1}{5}(1+1+1+1+1)$, and $\sigma_X^2 = 1 - \frac{1}{25} = \frac{24}{25}$.

   (c) Suppose that a different receiver samples the signal at time S, where

$$p_S(v) = \begin{cases} \frac{1}{N} & v = n\pi - \frac{\pi}{2}, \quad n \in \{1, \ldots, N\} \\ 0 & \text{otherwise,} \end{cases}$$

   where N is a constant positive integer. Let the measured signal sample be $Y = \sin(S)$. Find $E[Y]$, $E[Y^2]$, and σ_Y^2. Be sure to describe the outcome in the case when N is even, as well as the case when N is odd.

   Solution: When N is even, there are an equal number of $Y = 1$ and $Y = -1$ cases, so $E[Y] = 0$, $E[Y^2] = 1$, and $\sigma_Y^2 = 1$. When N is odd, $E[Y] = \frac{1}{N}$, $E[Y^2] = 1$, and $\sigma_Y^2 = 1 - \frac{1}{N^2}$.

2. **[Alice, Bob, and Chess]**
   Busses are often late. Bus 5A Amber and bus 9B Brown are late with probabilities $P(A) = 0.1$ and $P(B) = 0.25$, respectively. A reporter from the Daily Illini has observed that, whenever bus 5A is late, bus 9B is also late, i.e., $P(B|A) = 1.0$. The reporter suspects that Bob, the driver of bus 9B, and Alice, the driver of bus 5A, are secretly meeting to play chess once every few days, and that their chess games make them late.

   (a) Alice defends herself by saying that $P(AB)$ is only $\frac{1}{10}$; since $P(AB) = P(A)$, she argues, the events A and B must be independent. What's wrong with her argument?
   Solution: Independence is defined by $P(AB) = P(A)P(B)$, not by $P(AB) = P(A)$.

   (b) The reporter wants to find out how often Bob and Alice play chess. Let C be the event "Bob and Alice play chess." The reporter postulates that $P(A|C) = 1$ and $P(B|C) = 1$, i.e., a chess game always makes them late. She further postulates that, on days without chess, Bob and Alice are independent, i.e., $P(AB|C^c) = P(A|C^c)P(B|C^c)$. Under these assumptions (including the assumption $P(AB) = 0.1$ from part (a)), what is $P(C)$?

Solution: Bob and Alice can be late because of chess, or they can be late independently, thus $P(AB) = P(AB|C)P(C) + P(AB|C^c)P(C^c)$. The two postulates give us $P(AB) = P(C) + P(A|C^c)P(B|C^c)(1 - P(C))$. Similarly, $P(A^cB) = (1 - P(A|C^c))P(B|C^c)(1 - P(C))$ and $P(AB^c) = (1 - P(B|C^c))P(A|C^c)(1 - P(C))$. This gives us three equations for the three unknowns, $P(C)$, $(A|C^c)$, and $P(B|C^c)$. Solving yields $P(C) = 0.1$.

3. **[Independent and dependent events in a Karnaugh map]**
There is a well-dressed penguin drinking a martini at Boltini. Let A be the event "Penguin is wearing a bowtie," and let B be the event "Penguin knows how to differentiate e^x."

   (a) Suppose $P(A) = 2/3$, $P(B) = 1/5$, and the events A and B are independent. Draw a Karnaugh map of the experimental outcomes. Fill in the probabilities $P(AB)$, $P(AB^c)$, $P(A^cB)$, and $P(A^cB^c)$.

   Solution: Solution should be a Karnaugh map showing $P(AB) = 2/15$, $P(AB^c) = 8/15$, $P(A^cB) = 1/15$, and $P(A^cB^c) = 4/15$.

   (b) Suppose $P(A) = 2/3$, $P(B) = 1/5$, and $P(B|A) = 1/10$. Draw a Karnaugh map of the experimental outcomes. Fill in the probabilities $P(AB)$, $P(AB^c)$, $P(A^cB)$, and $P(A^cB^c)$.

   Solution: Solution should be a Karnaugh map showing $P(AB) = 1/15$, $P(AB^c) = 9/15$, $P(A^cB) = 2/15$, and $P(A^cB^c) = 3/15$.

4. **[The Bizarre Bernoulli]**
The Bernoulli distribution $p_X(u)$ is one of the only interesting probability distributions that is strictly bounded, on at least one side, by a one-standard-deviation bound, in the following sense. Recall that the Bernoulli distribution is defined by:

$$p_X(u) = \begin{cases} p & u = 1 \\ 1-p & u = 0 \\ 0 & \text{otherwise} \end{cases}$$

Prove the following statement: for every value of p, either $E[X] - \sigma_x \le 0$, or $E[X] + \sigma_x \ge 1$, or both, depending on the value of p.

(Note: this statement is interesting and worth proving because it implies that either $P\{X \ge E[X] - \sigma_X\} = 1$ or $P\{X \le E[X] + \sigma_X\} = 1$ or both, depending on the value of p. There are not many types of random variables that can be so tightly bounded.) **Solution:** $E[X] = p$, and $\sigma_x = \sqrt{p(1-p)}$. When $p \le 0.5$, $\sigma_X \ge p$, therefore $E[X] - \sigma_X \le p - p = 0$. When $p \ge 0.5$, $\sigma_X \ge (1-p)$, therefore $E[X] + \sigma_X \ge p + (1-p) = 1$.

5. **[Independent and Dependent Random Variables]**
Joe and Mary each flip a coin. Let X equal the total number of heads showing, thus $X \in \{0, 1, 2\}$.

   (a) Assume that both coins are fair, and that the coin flips are independent. Find $p_X(u)$.

   Solution: $p_X(0) = 1/4$, $p_X(1) = 1/2$, $p_X(2) = 1/4$.

   (b) Now suppose that Joe decides to impress Mary with his skills at magic: Joe waits until he can see whether Mary's coin is showing heads or tails, then spins his coin so that it tends to land the same way as Mary's. Specifically,

$$
\begin{aligned}
P(\text{Joe gets heads}|\text{Mary gets heads}) &= 0.75 \\
P(\text{Joe gets tails}|\text{Mary gets tails}) &= 0.75
\end{aligned}
$$

Find $p_X(u)$ under this new assumption.

Solution: The probability of two heads is $p_X(2) = (1/2)(3/4) = 3/8$. Likewise $p_X(0) = 3/8$, and $p_X(1) = 1/4$.