ECE 313: Problem Set 2: Solutions
Discrete random variables

1. **[Maximum and minimum values for probabilities]**

   (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$. Thus, $P(A \cup B) \leq 0.8$ with equality occurring when $P(A \cap B) = 0$.
   Similarly, $P(A \cup C) = P(A) + P(C) - P(A \cap C) \leq P(A) + P(C)$ But, $P(A) + P(C) > 1$ and so we conclude that $P(A \cup C) \leq 1$ with equality occurring when $P(A \cap C) = 0.3$.

   (b) Since $A \subset A \cup B$ and $B \subset A \cup B$, we know that $P(A) \leq P(A \cup B)$, $P(B) \leq P(A \cup B)$ and so $\max\{P(A), P(B)\} = 0.6 \leq P(A \cup B)$. Thus, $P(A \cup B) \geq 0.6$ with equality occurring when $B \subset A$. Similarly, $\max\{P(A), P(C)\} = 0.7 \leq P(A \cup C)$ with equality occurring when $A \subset C$.

   (c) Since $A \cap B \subset A$ and $A \cap B \subset B$, we have that $P(A \cap B) \leq P(A)$, $P(A \cap B) \leq P(B)$ and so $P(A \cap B) \leq \min\{P(A), P(B)\} = 0.2$. Thus, $P(A \cap B) \leq 0.2$ with equality occurring when $B \subset A$. Similarly, $P(A \cap C) \leq \min\{P(A), P(C)\} = 0.6$ with equality occurring when $A \subset C$.

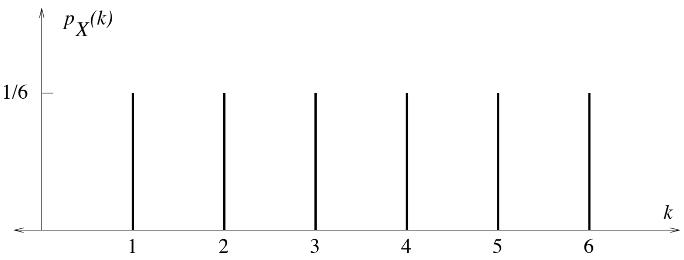
   (d) The smallest possible value of $P(A \cap B)$ is 0 when A and B are mutually exclusive. On the other hand, the smallest possible value of $P(A \cap C)$ is 0.3 in which case $P(A \cup C) = 1$ as noted in part (a).
   In summary, $0.6 \leq P(A \cup B) \leq 0.8$ where the minimum value occurs when $B \subset A$ (and thus $P(A \cap B) = P(B) = 0.2$ has a maximum value); while the maximum value 0.8 of $P(A \cup B)$ occurs when $P(A \cap B)$ has its minimum possible value 0.
   Similarly, $0.6 \leq P(A \cup C) \leq 1$ where the minimum value occurs when $A \subset C$ (and thus $P(A \cap C) = P(A) = 0.6$ has a maximum value); while the maximum value 1 of $P(A \cup C)$ occurs when $P(A \cap C)$ has its minimum possible value 0.3.

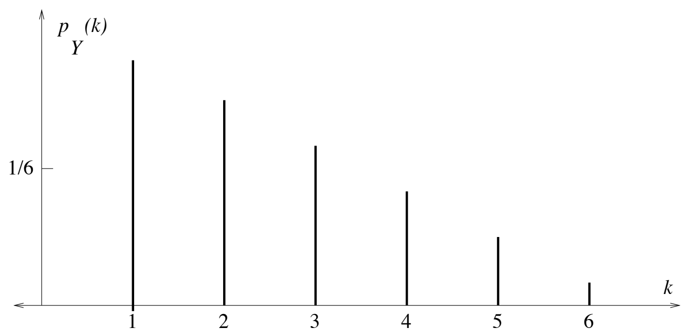
2. **[Mean and standard deviation]**

   (a) For $1 \leq k \leq 6$, there are six sample points with $X(i, j, \ell) = k$, so $p_X(k) = \frac{6}{36} = \frac{1}{6}$.

   (b) $E[X] = \frac{1+2+3+4+5+6}{6} = 3.5$. To compute σ_X we can first find $E[X^2] = \frac{1+2^2+3^2+4^2+5^2+6^2}{6}$ and then $\sigma_X = \sqrt{E[X^2] - E[X]^2} = 1.707825$.

   (c) $\{Y = 6\} = \{6, 6, 6)\}$,
   $\{Y = 5\} = \{(5, 6, 6), (6, 5, 6), (6, 6, 5), (6, 5, 5), (5, 6, 5), (5, 5, 6), (5, 5, 5)\}$, and so on. In general, for $1 \leq k \leq 6$, there are $1 + 3(6 - k) + 3(6 - k)^2$ sample points for which $Y = k$. Thus, $p_Y(k) = \frac{1+3(6-k)+3(6-k)^2}{216}$ for $1 \leq k \leq 6$.





(d) Using part (c), we find $E[Y] = 2.0417$ and $\sigma_Y = 1.1438$.

(e) $\sigma_Y < \sigma_X$. This is consistent with the sketches. Intuitively, the triangular shape of p_Y is more concentrated than the rectangular shape of p_X.

3. **[Die Roll]**

(a) There are three even numbered faces and three odd numbered faces. Each face is equally likely, so the probability that f_1 is even is 0.5.

(b) The sample space consists of a countably infinite number of outcomes, with the event $A_{(k,\ell)} =$"$f_1 = k$ and $f_2 = \ell$" consisting of all outcomes of the form $k\ell, kk\ell, kkk\ell, \ldots, k^n\ell, \ldots$ where, of course, $\ell \neq k$. Obviously,

$$P(A_{(k,\ell)}) = \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^3 + \left(\frac{1}{6}\right)^4 + \cdots = \left(\frac{1}{6}\right)^2 \left[1 + \frac{1}{6} + \left(\frac{1}{6}\right)^2 + \cdots\right] = \left(\frac{1}{6}\right)^2 \times \frac{1}{1-\frac{1}{6}} = \frac{1}{30}.$$

Thus, we can work with a much simpler model of a sample space

$$\Omega = \{(k,\ell) : k \neq \ell, \quad k = 1,\ldots,6, \quad \ell = 1,\ldots,6.\}$$

containing 30 equally likely outcomes. There are 6 outcomes that make up the event that both f_1 and f_2 are even (these are (2,4), (2,6), (4,2), (4,6), (6,2) and (6,4)). So, the probability that both f_1 and f_2 are even is $\frac{6}{30} = \frac{1}{5}$.

(c) The number of outcomes that make up the event that $f_1 + f_2 \leq 7$ is readily calculated. If $f_1 = 1$, then f_2 can be any of $\{2,3,4,5,6\}$. If $f_1 = 2$, then f_2 can be any of $\{1,3,4,5\}$. If $f_1 = 3$, then f_2 can be any of $\{1,2,4\}$. If $f_1 = 4$, then f_2 can be any of $\{1,2,3\}$. If $f_1 = 5$, then f_2 can be any of $\{1,2\}$. If $f_1 = 6$, then f_2 can be only be $\{1\}$. Thus, $f_1 + f_2 \leq 7$ if (f_1, f_2) is any of $5+4+3+3+2+1 = 18$ outcomes in Ω. So, the probability of this event is $\frac{18}{30} = 0.6$.

4. **[Defining Events]**

(a) $A \cup B \cup C$.

(b) $A^cB^cC^c$.

(c) ABC.

(d) $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$.

(e) $(A^c \cap B^c \cap C^c) \cup (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$

(f) $A \cap B \cap C^c$.

(g) $A \cup (A^c \cap B^c)$.