University of Illinois **Spring 2013**

ECE 313: Problem Set 1: Solutions
Axioms of probability and calculating the sizes of sets

1. **[Defining a set of outcomes]**

   (a) There are six games in all. We could order the games in any convenient way; for instance, the first game could be between teams 1 and 2, the second between teams 1 and 3, the third between teams 1 and 4, the fourth between teams 2 and 3, the fifth between 2 and 4 and the sixth between teams 3 and 4. Then one choice would be $\Omega = \{(w_1, w_2, w_3, w_4, w_5, w_6) : w_k \in \{0, 1\}\}$. Here w_k represents a loss (0) or a win (1) for the lower numbered team in game numbered k, for each $k = 1 \ldots 6$.

   (b) $2^6 = 64$, because there are two possible outcomes for each of the six games.

2. **[Possible probability assignments]**

   (This is one of many ways to get the answer.) Since, by De Morgan's law, the complement of $A \cup B$ is A^cB^c, the fact $P(A \cup B) = 0.5$ is equivalent to $P(A^cB^c) = 0.5$. Thus, we can fill in the Karnaugh diagram for A and B as shown:

	B^c	B
A^c	0.5	$0.3 - a$
A	a	0.2

   We filled in the variable a for $P(AB^c)$, and then, since the sum of the probabilities is one, it must be that $P(A^cB) = 0.3 - a$. The valid values of a are $0 \le a \le 0.3$, and $(P(A), P(B)) = (0.2 + a, 0.5 - a)$. So, in parametric form, the set of possible values of $(P(A), P(B))$ is $\{(0.2 + a, 0.5 - a) : 0 \le a \le 0.3\}$. Equivalent ways to write this set are $\{(u, v) : v = 0.7 - u \text{ and } 0.2 \le u \le 0.5\}$ or $\{(x, 0.7 - x) : 0.2 \le x \le 0.5\}$.

3. **[Grouping students into teams]**

   (a) One solution is the following. Four teams, numbered one through four, can be sequentially selected as follows. To begin, there are $\binom{12}{3}$ ways to choose team one. That leaves ine students, so for any choice of team one, there are $\binom{9}{3}$ ways to choose team two. That leaves $\binom{6}{3}$ ways to choose team three, and $\binom{3}{3} = 1$ way to choose team four. Thus, there are $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{4}{2}\binom{3}{3}$ ways to choose teams numbered one through five. But team numbers don't matter, and there are 4! ways to number the teams. So each team formation can be arrived at by the above procedure 4! ways. So the number of ways to form teams (with team numbers not mattering) is $\frac{\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{4!} = 15400$.
   A second solution is the following. There are 12! orderings of the students in the class. Given an ordering, we can pair the first student with the second and third to make the first team, pair the fourth with the fifth and sixth to make the second team, and so on. Because the ordering within teams and the ordering of the teams themselves does not matter, many permutations produce an equivalent formation of teams. We need to count, for one formation of teams, how many permutations produce that formation. The number is $6^4 4!$, because there are six ways to order the students within each team, and 4! ways to order the teams. So the total number of team formations with two students per team is 15400.

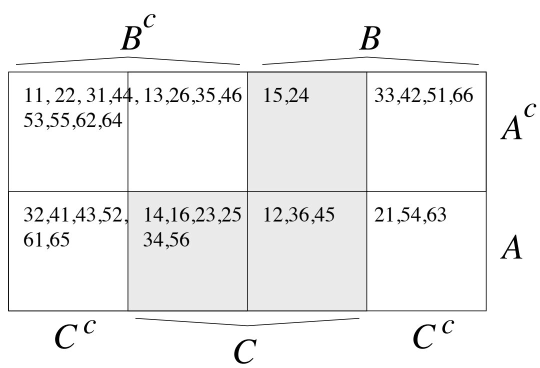
(b) As already found in part (a), there are 15400 ways to form teams with all teams of size three. So the only other possibility for team sizes are six tams of size two and two teams of size three and three teams of size two.

The first case is very similar to that in part (a). The number of teams is simply $\frac{\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{6!} = 10395$. You can also check your answer the second way: $\frac{12!}{2^6 6!} = 10395$.

Now for the second case. Five teams, numbered one through five, such that teams one and two each have three students and teams three, four and five each have two students, can be formed as follows. To begin, there are $\binom{12}{3}$ ways to choose team one. That leaves nine students, so for any choice of team one, there are $\binom{9}{3}$ ways to choose team two. That leaves $\binom{6}{2}$ ways to choose team three, and $\binom{4}{2}$ ways to choose team four and $\binom{2}{2}$ ways to choose team five. Thus, there are $\binom{12}{3}\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}$ ways to choose teams numbered one through five, such that teams one and two have three students each and teams three and four and five have two students each. But team numbers don't matter, so for a given formation of teams with two of size three and two of size two , there are two ways to decide which would be team one and which would be team two, and two ways to decide which would be team three and which would be team four. So each team formation can be arrived at by the above procedure in $2 \cot 6 = 12$ ways. So the number of ways to form teams (with team numbers not mattering) is $\frac{\binom{12}{3}\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{2 \cdot 6}$. This number simplifies to $\frac{10!}{(3!)^2 2^2 2^2} = 138,600$.

Therefore, the total number of team formations with each team having either two or three students is $10395 + 138600 + 15400 = 164395$.

4. **[A Karnaugh map for three events]**



(a)

(b) The set $(A \cup B)C$ corresponds to the shaded region in the Karnaugh map. Nine of the 36 outcomes are in the shaded region, so $P((A \cup B)C) = \frac{11}{36}$.

5. **[Cards]**

(a) There are $\binom{16}{5} = 4128$ ways of cards in the event $E1$. Normalizing by the number of all outcomes $\binom{52}{5}$ we get the probability of $E1$ to be $\frac{\binom{16}{5}}{\binom{52}{5}}$.

(b) Now $E2$ is the union of two events, $E4$ and $E3$ which are defined as follows:

- $E3$ is the event of having four of the cards to be A,K,Q,J of the same suit;
- $E4$ is the event of having all five cards of the same suit.

The number of outcomes in $E3$ is $4 \cdot 48 = 192$ (because, there are four suits to choose from for the A,K,Q,J and the fifth card can be any of the remaining 48 cards). The number of outcomes in $E4$ is $\binom{13}{4} \cdot 4$. The number of outcomes common to $E1$ and $E3$ is $4 \cdot 9 = 36$. So we conclude that the number of outcomes in $E2 = E4 \cup E3$ is 5148 + 192 - 36 = 5304. So the probability of $E2$ is $\frac{5304}{\binom{52}{5}}$.