University of Illinois **Spring 2013**

ECE 313: Problem Set 13
Covariance and MMSE Estimation

Due: Wednesday April 24, at 6 p.m..
Reading: ECE 313 notes, Chapter 4.7-4.9

1. **[Some moments for a random rectangle]**
Let $A = XY$ denote the area and $L = 2(X+Y)$ the length of the perimeter, of a rectangle with length X and height Y, such that X and Y are independent, and uniformly distributed on the interval $[0, 1]$.
   (a) Find $E[A]$ and $E[L]$.
   (b) Find $\text{Var}(A)$. (Hint: Find $E[A^2]$ first.)
   (c) Find $\text{Var}(L)$.
   (d) Find $\text{Cov}(A, L)$. (Hint: Find $E[AL]$ first.)
   (e) Find the correlation coefficient, $\rho_{A,L}$. (Hint: Should be less than, but fairly close to, one. Why?)

2. **[Jointly Distributed Random Variables]**
Let the random variables X and Y be such that $E[X] = 1$, $E[Y] = 4$, $Var(X) = 4$, $Var(Y) = 9$, and $\rho = 0.1$. Let $W = 3X + Y + 2$. Find $E[W]$ and $Var(W)$.

3. **[Linear Minimum Mean Square Error Estimation]**
Suppose that the value of Y is to be estimated in terms of the uncorrelated random variables X_1 and X_2 by the linear predictor $g(X_1, X_2) = a + bX_1 + cX_2$. Determine the values for a, b and c that minimize $E[(Y - (a + bX_1 + cX_2))^2]$. Express your answer in terms of the means and variances of X_1 and X_2 and the covariances $Cov(X, Y_1)$ and $Cov(X, Y_2)$.

4. **[Covariance I]**
Consider random variables X and Y on the same probability space.
   (a) If $\text{Var}(X + 2Y) = 40$ and $\text{Var}(X - 2Y) = 20$, what is $\text{Cov}(X, Y)$?
   (b) In part (a), determine $\rho_{X,Y}$ if $\text{Var}(X) = 2 \cdot \text{Var}(Y)$.

   The next two parts are independent of parts (a) and (b), and of each other. In particular, the numbers from part (a) are not to be assumed.

   (c) If $\text{Var}(X + 2Y) = \text{Var}(X - 2Y)$, are X and Y uncorrelated ?
   (d) If $\text{Var}(X) = \text{Var}(Y)$, are X and Y uncorrelated?

5. **[Covariance II]**
Rewrite the expressions below in terms of $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Var}(Z)$, and $\text{Cov}(X, Y)$.
   (a) $\text{Cov}(3X + 2, 5Y - 1)$
   (b) $\text{Cov}(2X + 1, X + 5Y - 1)$.
   (c) $\text{Cov}(2X + 3Z, Y + 2Z)$ where Z is uncorrelated to both X and Y.

6. **[Covariance III]**
Random variables X_1 and X_2 represent two observations of a signal corrupted by noise. They have the same mean μ and variance σ^2. The *signal-to-noise-ratio* (SNR) of the observation X_1 or X_2 is defined as the ratio $SNR_X = \frac{\mu^2}{\sigma^2}$. A system designer chooses the averaging strategy, whereby she constructs a new random variable $S = \frac{X_1 + X_2}{2}$.
   (a) Show that the SNR of S is twice that of the individual observations, if X_1 and X_2 are uncorrelated.

(b) The system designer notices that the averaging strategy is giving $SNR_S = (1.5)SNR_X$. She correctly assumes that the observations X_1 and X_2 are correlated. Determine the value of the correlation coefficient $\rho_{X_1 X_2}$.

(c) Under what condition on ρ_{X_1,X_2} can the averaging strategy result in an SNR_S that is as high as possible?