ECE 313: Problem Set 9
Gaussian Random Variable, Functions of a Random Variable

Due: Wednesday, March 27 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.6-3.8.

1. **[Gaussian CDF]**
   Let $\mathbb{X}$ denote a unit Gaussian random variable. Its CDF is $\Phi(u)$.

   (a) What is the derivative of $\exp(-u^2/2)$? Use this result to compute $\mathsf{E}[\|\mathbb{X}\|]$.

   (b) $Q(x) = 1 - \Phi(x)$ is the *complementary* CDF. A useful bound is $Q(x) \leq \frac{1}{2}\exp(-x^2/2)$ for $x \geq 0$. Derive this bound by first proving that $t^2 - x^2 > (t-x)^2$ for $t > x > 0$ and then applying this to

   $$\exp(x^2/2)Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2-x^2}{2}\right)dt.$$

2. **[Table of Q functions]**
   The width of a wire on a circuit board is modeled as a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

   (a) Wires that fail to meet the quality requirement that the width be within the range 0.9 ± 0.005 microns are said to be defective. What percentage of wires are defective?

   (b) A new manufacturing process that produces smaller variations in wires is to be designed so that no more than 1 defective wires occur in 100. What is the maximum value of σ for the new process if the new process achieves the goal?

3. **[Function of a Random Variable]**
   The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathcal{V}$ with pdf

   $$f_{\mathcal{V}}(u) = 0.5\exp(-|u|), -\infty < u < \infty.$$

   Then, the current $\mathcal{I}$ is also a continuous random variable.

   (a) What values can $\mathcal{I}$ take on?

   (b) Find the CDF of $\mathcal{I}$.

   (c) Find the pdf of $\mathcal{I}$.

4. []
   A signal $\mathbb{X}$ is modeled as a unit (standard) Gaussian random variable. For some applications, however, only the quantized value $\mathbb{Y}$ (where $\mathbb{Y} = \alpha$ if $\mathbb{X} > 0$ and $\mathbb{Y} = -\alpha$ if $\mathbb{X} \leq 0$) is used. Note that $\mathbb{Y}$ is a *discrete* random variable.

   (a) What is the pmf of $\mathbb{Y}$?

   (b) The *squared error* in representing $\mathbb{X}$ by $\mathbb{Y}$ is $\mathbb{Z} = \begin{cases} (\mathbb{X}-\alpha)^2, & \text{if } \mathbb{X} > 0, \\ (\mathbb{X}+\alpha)^2, & \text{if } \mathbb{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathbb{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathbb{Z}]$. Use LOTUS to calculate $\mathsf{E}[\mathbb{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathbb{Z}]$.

(c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathbb{X}$ to the nearest integer $\mathbb{W}$ in the range 3 to +3. Thus, $\mathbb{W} = 3$ if $\mathbb{X} \geq 2.5$, $\mathbb{W} = 2$ if $1.5 \leq \mathbb{X} < 2.5$, $\mathbb{W} = 1$ if $0.5 \leq \mathbb{X} < 1.5$, $\cdots$, $\mathbb{W} = -3$ if $\mathbb{X} < -2.5$. Note that $\mathbb{W}$ is also a discrete random variable. Find the pmf of $\mathbb{W}$.

5. []
Consider a sphere whose radius is a random variable $\mathcal{R}$ with pdf $f_{\mathcal{R}}(u) = 2u$, $0 < u < 1$, and 0 otherwise.

(a) What is the average radius of the sphere?

(b) What is the average volume?

(c) What is the average surface area?

(d) If a sphere of average radius is called an *average sphere*, then does an average sphere have the average volume? Does it have the average surface area?

6. []
$\mathbb{X}$ denotes a *uniform* random variable with mean 1 and variance 3. Find the pdf of $\mathbb{Y} = |\mathbb{X}|$.