University of Illinois Spring 2013

ECE 313: Final Exam

Friday May 3, 2013
8:00 a.m. — 11:00 a.m.

Name: (in BLOCK CAPITALS) ________________________________

University ID Number: ____________________

Signature: ________________________________

Section: ☐ E, 9:00 a.m. ☐ C, 10:00 a.m. ☐ D, 11:00 p.m. ☐ F, 1:00 p.m.

Instructions

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75). Irrational numbers such as e and π should be expressed using their symbols; you should not try to substitute any approximation.

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 10 points ________	6. 25 points ________
2. 21 points ________	7. 25 points ________
3. 19 points ________	8. 25 points ________
4. 20 points ________	9. 30 points ________
5. 25 points ________	10. 25 points ________
	Total (225 points) ________

Table of Probability Distributions			
Name	pmf/pdf	Mean	Variance
Bernoulli	$p_X(k) = \begin{cases} p & k=1 \\ 1-p & k=0 \\ 0 & \text{otherwise} \end{cases}$	p	$p(1-p)$
Binomial	$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$	np	$np(1-p)$
Geometric	$p_X(k) = \begin{cases} p(1-p)^{k-1} & 1 \leq k \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$
Negative Binomial	$p_X(k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r} & r \leq k \\ 0 & \text{otherwise} \end{cases}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson	$p_X(k) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!} & 0 \leq k \\ 0 & \text{otherwise} \end{cases}$	λ	λ
Exponential	$f_X(u) = \begin{cases} \lambda e^{-\lambda u} & 0 \leq u \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang	$f_X(u) = \begin{cases} \frac{\lambda^r u^{r-1} e^{-\lambda u}}{(r-1)!} & 0 \leq u \\ 0 & \text{otherwise} \end{cases}$	$\frac{r}{\lambda}$	$\frac{r}{\lambda^2}$
Uniform	$f_X(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{otherwise} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Gaussian	$f_X(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}$	μ	σ^2

1. **[10 points]** Goldsmith is a coin collector. His coin pouch contains g (American) Golden Eagles and m (Canadian) Maple Leafs. Goldsmith is picking up coins at random from the pouch. In order to express a concept like "M choose N" in this problem, please use the notation $\binom{M}{N}$.

   (a) **[5 points]** Suppose that Goldsmith picks up a total of k coins (coins once picked are not put back). What is the probability of picking up x Golden Eagles? Suppose that $x \leq k$.

   $P\{\text{picking up } x \text{ Golden Eagles}\} =$

   (b) **[5 points]** Now suppose that all k coins are returned to the pouch, and Goldsmith picks up a total of n coins this time. Let A be the event that among the selected n coins, exactly 3 Golden Eagles are included that were also selected in the first picking? Find $P(A)$. Suppose that $x \geq 3$ and $n \geq 3$.

   $P(A) =$

2. **[21 points]** 3 points per correct answer. In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score. There is no need to justify your answer.

   (a) A, B, C are three events such that $0 < P(A) < 1$, $0 < P(B) < 1$ and $0 < P(C) < 1$.

TRUE	FALSE	
☐	☐	$P(A\|B) + P(A\|B^c) = P(A)$
☐	☐	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$.
☐	☐	$P(A \cup B \cup C) \geq P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$.

   (b) Consider a binary hypothesis testing problem where the prior probability of hypothesis H_0 is π_0 and the prior probability of hypothesis H_1 is π_1. Denote the probabilities of false alarm and missed detection for the ML decision rule by P_{FA}^{ML} and P_{MD}^{ML}, respectively. Similarly, denote the probabilities of false alarm and missed detection for the MAP decision rule by P_{FA}^{MAP} and P_{MD}^{MAP}, respectively.

TRUE	FALSE	
☐	☐	$P_{FA}^{ML} + P_{MD}^{ML} = 1$.
☐	☐	$P_{FA}^{ML}\pi_0 + P_{MD}^{ML}\pi_1 = 1$.
☐	☐	If $P_{FA}^{ML} \cdot \pi_0 + P_{MD}^{ML} \cdot \pi_1 = P_{FA}^{MAP} \cdot \pi_0 + P_{MD}^{MAP} \cdot \pi_1$ then $\pi_0 = \pi_1$.
☐	☐	If $\pi_0 = 0.5$ then $P_{MD}^{ML} = P_{MD}^{MAP}$.

3. [**19 points**] A quiz show contestant answers each question incorrectly with probability $\frac{1}{3}$. The number of questions asked in any show is a random variable $\mathbb{Y}$, geometrically distributed with parameter $\frac{1}{10}$.

(a) [**5 points**] Let A be the event that the contestant gives wrong answers to each of the questions asked in a show. Find $P(A)$.

$P(A) =$

(b) [**5 points**] Let B be the event that the contestant was asked only 2 questions in a show. Find $P[B|A]$.

$P(B|A) =$

PROBLEM 3, CONTINUED

(c) [**3 points**] There are 3 points for a correct answer and a negative 3 points for an incorrect answer. Zero points for an omitted answer. Let $\mathbb{X}$ be the number of questions correctly answered by the contestant in a show. State TRUE or FALSE. There is no need to justify your answer.

TRUE	FALSE	
☐	☐	$\rho_{\mathbb{X},\mathbb{Y}} > 0$.

(d) [**6 points**] Show that $E[\mathbb{X}] \leq 10$.

4. **[20 points]** The parts of this question are unrelated to each other.

(a) **[6 points]** Let N be the number of Heads showing in 1800 independent tosses of a biased coin, with probability of heads equal to $\frac{1}{3}$. Approximate $P(N \leq 580)$, using the $Q(\cdot)$ function, as well as you can.

$\Pr\{N \leq 580\} =$

(b) **[5 points]** Two independent, uniformly distributed random variables $\mathbb{X}$ and $\mathbb{Y}$, both with mean 0 and variance 12, are summed up to yield a random variable $\mathbb{Z}$. Sketch the pdf of $\mathbb{Z}$, taking care to numerically label the important points in the sketch.

PROBLEM 4, CONTINUED

(c) **[3 points]** Two random variables $\mathbb{X}$ and $\mathbb{Y}$ are jointly distributed with pdf $f_{\mathbb{X},\mathbb{Y}}(u,v) = C$ for $|u| + |v| \leq 1$. Further, $f_{\mathbb{X},\mathbb{Y}}(u,v) = 0$, otherwise. Find the value of C.

$C =$

(d) **[3 points]**

TRUE	FALSE	
☐	☐	$\mathbb{X}$ and $\mathbb{Y}$ are independent.

(e) **[3 points]**

TRUE	FALSE	
☐	☐	$\mathbb{X}$ and $\mathbb{Y}$ are uncorrelated.

5. **[25 points]** Suppose X and Y are independent random variables with $\mu_X = 3$, $\mu_Y = 0$, $\text{Var}(X) = 7$ and $\text{Var}(Y) = 9$. Let $Z = X + Y$ and $U = X - Y$.

   (a) **[10 points]** Find the linear estimator $L^*(Z)$ of X based on Z with the minimum MSE.

$L^*(Z) =$

   (b) **[9 points]** Find $\rho_{U,Z}$.

$\rho_{U,Z} =$

PROBLEM 5, CONTINUED

(c) [**6 points**] V is some random variable, with mean zero, that is jointly distributed with X. Given V, it is known that the best linear estimator of X is a constant, say c. Find c and $Cov(X, V)$.

6. **[25 points]** You are in Monte Carlo, watching a roulette wheel. The roulette wheel has four overlapping events painted on it: event A, event B, event C, and event D. Through all parts of this problem, $P(A) = \frac{1}{2}$.

(a) **[4 points]** $P(AB) = \frac{1}{6}$. Events A and B are independent. $P(A^c B^c)$ is defined to be the event that neither A nor B occurs. What is $P(A^c B^c)$?

$$P(A^c B^c) = \boxed{}$$

(b) **[4 points]** $P(C) = \frac{1}{2}$, and $P(AC) = \frac{1}{6}$. What is $P(A^c C^c)$?

$$P(A^c C^c) = \boxed{}$$

(c) **[8 points]** $P(A|D) = \frac{2}{3}$. What are the minimum and maximum possible values of $P(D)$?

$$\boxed{} < P(D) \le \boxed{}$$

(d) **[9 points]** When A occurs, the dealer gives you \$10. When A^c occurs, you lose \$5. Assuming that you have \$100 when you start the game, what is the probability that you have more than \$182 after playing the game 10 times?

$$\Pr\{100 + \text{Winnings} \ge 182\} = \boxed{}$$

7. **[25 points]** The tiny village of Jensbro is connected to the outside world by only four (4) telephone circuits; if all 4 circuits are busy, then anybody else who tries to make a call will get a busy signal. Fortunately, nobody from the outside world ever calls Jensbro. Unfortunately, the residents of Jensbro place outbound calls unpredictably: each call is placed independently of any other calls, at an average rate of λ calls per minute. It is therefore extremely important to determine the value of λ.

   (a) **[9 points]** The telephone engineer, Dr. Erlang, devises the following experiment. At exactly 9:00 every day, he clears all 4 telephone circuits (by rudely disconnecting any ongoing conversations). He then measures the number of minutes, T, until four more phone calls have been placed. Find the maximum likelihood estimate of λ given T.

   $$\hat{\lambda}_{ML} = \boxed{}$$

   (b) **[8 points]** Suppose that Dr. Erlang has done the experiment described in part (a), and has discovered that $\lambda = 0.2$. Suppose, further, that each call lasts exactly ten minutes. If fewer than 4 phone calls have been placed in the most recent 10 minutes, a phone call gets connected; if not, the caller hears a busy signal. What is the probability that any given call gets connected?

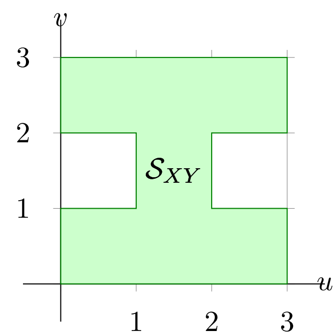
   $$\Pr\{3 \text{ or fewer phone calls in 10 minutes}\} = \boxed{}$$

PROBLEM 7, CONTINUED

(c) [**8 points**]Define X to be the time of the first phone call of the day (the number of minutes past 9:00 at which it occurs). Define Y to be the time of the second phone call of the day (number of minutes past 9:00 at which it occurs). Assume that $\lambda = 0.2$. Find the joint pdf $f_{X,Y}(u, v)$.

8. **[25 points]** Random variables X and Y are uniformly distributed in the I-shaped region $\mathcal{S}_{XY}$ shown below, thus:

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{7} & 0 \le v \le 1,\ 0 \le u \le 3 \\ \frac{1}{7} & 1 < v \le 2,\ 1 \le u \le 2 \\ \frac{1}{7} & 2 < v \le 3,\ 0 \le u \le 3 \\ 0 & \text{otherwise} \end{cases}$$



(a) **[8 points]** Find the marginal pdf $f_X(u)$ as a one-dimensional function of u. You may express your answer in an equation using cases, or you may sketch it; if you sketch it, be sure that your sketch clearly shows the value of $f_X(u)$ at each u.

(b) **[8 points]** Find the conditional pdf $f_{Y|X}(v|\frac{1}{2})$ as a one-dimensional function of v. You may express your answer in an equation using cases, or you may sketch it; if you sketch it, be sure that your sketch clearly shows the value of $f_{Y|X}(v|\frac{1}{2})$ at each v.

(c) **[9 points]** Find the CDF $F_{X,Y}(u, \frac{1}{2})$ as a one-dimensional function of u. You may express your answer in an equation using cases, or you may sketch it; if you sketch it, be sure that your sketch clearly shows the value of $F_{X,Y}(u, \frac{1}{2})$ at each u.

9. **[30 points]** At the World's End Cafe, the price of a brownie is a random variable $B \sim \text{Uniform}(2, 4)$ (meaning that each customer is charged a different price; the price for any given customer is uniformly distributed between \$2 and \$4). The price of a glass of milk is a random variable $M \sim \text{Uniform}(0, 1)$. Alice wants to buy a brownie and a glass of milk; the price she will pay is random variable $P = B + M$.

   (a) **[10 points]** Assume M and B are independent; find the pdf $f_P(w)$. You may express your answer in an equation using cases, or you may sketch it; if you sketch it, be sure that your sketch clearly shows the value of $f_P(w)$ at each w.

   (b) **[10 points]** Now suppose that M and B are NOT independent, but that you do not know the exact form of the joint pdf $f_{B,M}(u, v)$; instead, all you know is that they are uniform as stated above, and that their correlation coefficient is $\rho_{MB} = -\frac{1}{2}$. Find the variance $\text{Var}(P)$.

$$\text{Var}(P) = \boxed{}$$

PROBLEM 9, CONTINUED

(c) **[10 points]** Consider two other random variables, X and Y, that are NOT independent. Instead, they are distributed as

$$f_X(u) = \begin{cases} \frac{1}{2} & -1 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(v|u) = \begin{cases} \frac{1+uv}{2} & -1 \le v \le 1, \quad -1 \le u \le 1 \\ 0 & -1 \le u \le 1, \quad v \notin [-1,1] \\ \text{undefined} & u \notin [-1,1] \end{cases}$$

Define the random variable $Z = X + Y$. Find the probability density $f_Z(0)$, that is, the numerical value of $f_Z(w)$ at the particular instance $w = 0$.

$$f_Z(0) = \boxed{}$$

10. **[25 points]** Bob is an untalented martial artist. In order to earn his green belt, he must demonstrate that he can break boards by kicking them. He is not very good at breaking boards; each time he tries, he succeeds with a probability of only $p = \frac{1}{3}$, independently of his success on any other trial.

   (a) **[6 points]** In order to pass his test, Bob must break two boards. He is allowed only four attempts. What is the probability that he will pass his test?

   (b) **[6 points]** Bob's martial arts instructor decides to make the test easier to pass: Bob is allowed to keep trying until he manages to break two boards. Let T be the number of times Bob attempts to break a board, including both successful attempts, and also including all of his failures. What is the pmf of random variable T?

PROBLEM 10, CONTINUED

(c) [**6 points**] Bob's arch-enemy, Alice, makes the following wager:

- Bob will break his first board on the first attempt.
- On the next three attempts (attempts numbered 2 through 4), Bob will fail.
- On attempt number 5, Bob will finally break the second board.

What is the probability that Alice wins her wager?

(d) [**7 points**] Let F be the number of attempts up to and including Bob's first success—the first attempt that breaks a board. Let T be the number of attempts up to and including Bob's second success. Find the conditional pmf $p_{T|F}(t|f)$.