University of Illinois Spring 2013

ECE 313: Hour Exam II

Monday April 15, 2013
7:00 p.m. — 8:15 p.m.

1. (a)
$$E[X] = 2\lambda_A = 6$$

(b)
$$\Pr\{T > 1\} = \int_1^\infty 3e^{-3t}dt = e^{-3}$$

(c) In a Poisson process, non-overlapping intervals of time are independent, thus
$$E[T|T > 1] = 1 + E[T] = \frac{4}{3}$$

(d) The rates of the two Poisson processes add, so the total process has $\lambda = \lambda_A + \lambda_B = 8$, and
$$p_Y(2) = \frac{8^2 e^{-8}}{2!} = 32e^{-8}$$

2. (a)
$$\begin{aligned} \mathsf{E}[\mathbb{X}] &= \int_{-\infty}^{\infty} u f_{\mathbb{X}}(u)\, du \\ &= 0 \end{aligned}$$
since the function $u \cdot f_{\mathbb{X}}(u)$ is an odd function around the origin.

(b) $\mathbb{Y}^2 = 9\mathbb{X}^2 + 24\mathbb{X} + 16$. So $\mathsf{E}[\mathbb{Y}^2] = 9\mathsf{E}[\mathbb{X}^2] + 16 = 916$.

(c) The statement is true.

(d)
$$P[|\mathbb{X}| \leq 10] = 1 - 2P[\mathbb{X} > 10] = 1 - 2Q(1).$$

(e) $\mathbb{Z}$ is a Binomial random variable with mean 50 and variance 25. By the DeMoivre-Laplace approximation $P[\mathbb{Z} \leq 40]$ is closely approximated by $P[\tilde{\mathbb{X}} \leq 40.5]$ where $\tilde{\mathbb{X}}$ is a Gaussian random variable also with mean 50 and variance 25. Now the distributions of $\mathbb{X}$ and $2\tilde{\mathbb{X}} - 100$ are the same. So $P[\tilde{\mathbb{X}} \leq 40.5] = P[\mathbb{X} \leq -19] = P[\mathbb{X} \geq 19]$.

3. (a) $\Gamma_0 = \{0 \leq u \leq 2\}$.

   (b) The corresponding $P_{\text{MD}} = 0.25$.

   (c) Since we cannot afford to make a mistake whenever $\mathbb{X}$ is in the support of the conditional pdf, conditioned on H_1, the decision region for H_1, Γ_1 must contain the support $\{1 \leq u \leq 4\}$. To minimize the probability of false alarm, we need to pick Γ_0 as large as possible. The largest possible decision region for H_0 is: $\Gamma_0 = \{0 \leq u < 1\}$. The corresponding probability of false alarm is 0.5.

   (d) Now $P_{\text{FA}} = 1 - p$ and $P_{\text{MD}} = p$. The average probability of error is $P_e = \pi_0(1-p) + \pi_1 p$, which is minimized when $p = 0$ since $\pi_1 > \pi_0$. This should make intuitive sense: since H_1 is more likely than H_0, without any other information about them the natural decision rule would be to pick H_1.

4. (a) X and Y are not independent. Acceptable reasons include (1) the support is not a product set, (2) the profile $f_{X,Y}(u,v)$ changes shape as one sweeps across the graph, (3) $\frac{f_{X,Y}(a,d)}{f_{X,Y}(a,c)} \neq \frac{f_{X,Y}(b,d)}{f_{X,Y}(b,c)}$ for any four points of which three lie inside the support, and one lies outside, (4) the pdf can not be factored into marginals.

   (b) The sketch should show $f_{X|Y}(u|1)$ uniform between $u = -0.5$ and $u = 1.5$, with a density of $\frac{1}{2}$ in this region.

   (c) The sketch should show $f_X(u)$ triangular between $u = -2$ and $u = 2$, with a peak value of $f_X(0) = 0.5$.

   (d) The region of interest is $\{X \leq 0, Y \leq 1\}$. The resulting integral is $F_{X,Y}(0,1) = \frac{15}{32}$.