ECE 313: Problem Set 11: Solutions
Independent RVs; Sums of RVs

1. **[Independent Random Variables]**

(a) Independent. This is the product of the pmf of two independent binomial random variables.

(b) Not Independent. There are two ways to see this:

(1) The fact that they're not independent is given away by the support: the support of Y depends on the values of X.

(2) Alternatively, one can observe that there is no way to factor $p_{X,Y}(j,k)$ into a term that is independent of k, times a term independent of j. For example, one could factor it as follows:

$$p_{X,Y}(j,k) = \left(\frac{n!}{j!}p^j(1-2p)^{n-j}\right)\left(\frac{1}{k!}p^k\left(\frac{1}{1-2p}\right)^k\right)\left(\frac{1}{(n-j-k)!}\right),$$

but the last term on the right-hand side, $1/(n-j-k)!$, can't be factored into terms dependent only on j and k, therefore X and Y are not independent.

(c) Independent. This joint pdf is the product of two independent Gaussian pdfs.

(d) Independent. This pdf can be factored into a term dependent on u, and a term dependent on v.

(e) Not Independent. There are a few ways to see this:

(1) $f_{X,Y}(u,v) \neq f_X(u)f_Y(v)$ for any possible choice of $f_X(u)$ and $f_Y(v)$. This can be proven by noticing that $f_{X,Y}(0,0)f_{X,Y}(1,1) \neq f_{X,Y}(0,1)f_{X,Y}(1,0)$.

(2) $f_{Y|X}(v|u) \neq f_Y(v)$. This can be proven without explicitly calculating the marginal and conditional pdfs if one realizes that, for any particular value of u, $f_{Y|X}(v|u) = f_{X,Y}(u,v)/f_X(u)$, therefore as a function of v, $f_{Y|X}(v|u) \propto f_{X,Y}(u,v)$. Notice that $f_{X,Y}(0,v)$ has the shape $0.5\lambda^2 e^{-2\lambda|v|}$, but $f_{X,Y}(1,v)$ is constant in the range $-1 \leq v \leq 1$. Since $f_{X,Y}(u,v)$ as a function of v has different shapes at $u=0$ and $u=1$, it follows necessarily that $f_{Y|X}(v|u)$ has different shapes at $u=0$ and $u=1$, therefore $f_{Y|X}(v|u) \neq f_Y(v)$.

(f) This pdf can be factored, $f_{X,Y}(u,v) = f_X(u)f_Y(v)$, where

$$f_X(u) = \begin{cases} \frac{1}{6} & 1 \leq u \leq 4 \\ \frac{1}{2} & 5 \leq u \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(v) = \begin{cases} \frac{1}{2} & 1 \leq v \leq 2 \\ \frac{1}{6} & 3 \leq v \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

2. **[Conditional Distributions]**

(a) X and Y are independent, so $p_{Y|X}(k|j) = p_Y(k)$:

$$p_{Y|X}(k|j) = \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}, \quad 0 \le j \le n,\ 0 \le k \le n$$

(b) X and Y are not independent. In general, it is necessary first to find $p_X(j)$ before finding $p_{Y|X}(k|j)$:

$$\begin{aligned}
p_X(j) &= \sum_{k=0}^{n-j} \frac{n!}{j!k!(n-j-k)!}p^{j+k}(1-2p)^{n-j-k}, \quad 0 \le j \le n && (1)\\
&= \sum_{k=0}^{n-j} \left(\binom{n}{j}p^j\right)\left(\binom{n-j}{k}p^k(1-2p)^{n-j-k}\right), \quad 0 \le j \le n && (2)\\
&= \binom{n}{j}p^j(1-p)^{n-j}, \quad 0 \le j \le n && (3)
\end{aligned}$$

Thus X has the binomial pmf with parameters n and p. For j fixed with $0 \le j \le n$ the following holds for $0 \le k \le n-j$:

$$p_{Y|X}(k|j) = \frac{p_{X,Y}(j,k)}{p_X(j)} = \binom{n-j}{k}p^k\frac{(1-2p)^{n-j-k}}{(1-p)^{n-j}} = \binom{n-j}{k}\left(\frac{p}{1-p}\right)^k\left(\frac{1-2p}{1-p}\right)^{n-j-k}$$

Thus, the conditional distribution of Y given X is the binomial distribution with parameters $n-j$ and $\frac{p}{1-p}$.

(c) X and Y are independent, so $f_{Y|X}(v|u) = f_Y(v)$:

$$f_{Y|X}(v|u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}, \quad -\infty < u < \infty,\ -\infty < v < \infty$$

(d) X and Y are independent, so $f_{Y|X}(v|u) = f_Y(v)$:

$$f_{Y|X}(v|u) = \frac{\lambda}{2}e^{-\lambda|v|}, \quad -\infty < u < \infty,\ -\infty < v < \infty$$

(e) X and Y are not independent, therefore it is necessary first to find the marginal pdf $f_X(u)$. First, notice that

$$|u+v| = \begin{cases} |u|+v & v > -|u| \\ |u|-v & v < -u \end{cases}, \quad |u-v| = \begin{cases} v-|u| & v > |u| \\ |u|-v & v < u \end{cases}$$

By symmetry,

$$\begin{aligned}
f_X(u) &= \int_{-\infty}^{\infty} f_{X,Y}(u,v)dv && (4)\\
&= \int_{-|u|}^{|u|} \frac{\lambda^2}{2}e^{-2\lambda|u|}dv + 2\int_{|u|}^{\infty}\frac{\lambda^2}{2}e^{-2\lambda v}dv && (5)\\
&= \left(\frac{\lambda}{2} + \lambda^2|u|\right)e^{-2\lambda|u|} && (6)
\end{aligned}$$

Therefore

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)} = \begin{cases} \frac{\lambda}{1+2\lambda|u|} & |v| \le |u| \\ \left(\frac{\lambda}{1+2\lambda|u|}\right)e^{-2\lambda|v|} & |v| \ge |u| \end{cases}$$

(f) X and Y are independent, so $f_{Y|X}(v|u) = f_Y(v)$:

$$f_{Y|X}(v|u) = \begin{cases} \frac{1}{2} & 1 \le v \le 2 \\ \frac{1}{6} & 3 \le v \le 6 \\ 0 & \text{other values of } v \end{cases}$$

3. **[Poisson Processes and Their Kin]**

(a)

$$\begin{aligned} f_{T_2}(t) &= f_{U_1}(t) * f_{U_2}(t) && (7) \\ &= \int_0^t f_{U_1}(u) f_{U_2}(t-u) du && (8) \\ &= \lambda^2 \int_0^t e^{-\lambda u} e^{-\lambda(t-u)} du && (9) \\ &= \lambda^2 e^{-\lambda t} \int_0^t du && (10) \\ &= \lambda^2 t e^{-\lambda t} && (11) \end{aligned}$$

(b)

$$\begin{aligned} f_{T_2}(t) &= f_{U_1}(t) * f_{U_2}(t) && (12) \\ &= \int_b^{t-b} f_{U_1}(u) f_{U_2}(t-u) ds && (13) \\ &= \lambda^2 \int_b^{t-b} e^{-\lambda(u-b)} e^{-\lambda(t-u-b)} du && (14) \\ &= \lambda^2 e^{-\lambda(t-2b)} \int_b^{t-b} du && (15) \\ &= \lambda^2 (t-2b) e^{-\lambda(t-2b)}, \quad t \ge 2b && (16) \end{aligned}$$

An alternative solution could be given without computation. Changing a random variable by adding a constant $b > 0$ to it yields a random variable with pdf obtained by shifting the original pdf to the right by b. So the new pdf of $S + T$ can be obtained by shifting the pdf from part (b) to the right by $2b$.

(c) This is no longer a simple convolution, but the general formula in Eq. (4.15) of the notes still holds:

$$\begin{aligned} f_{T_2}(t) &= \int_0^t f_{U_1,U_2}(u, t-u) du && (17) \\ &= \int_0^t f_{U_1}(u) f_{U_2|U_1}(t-u|u) du && (18) \\ &= 0.5\lambda^2 \int_0^t e^{-\lambda u} e^{-\lambda(t-u)} du + 0.5\lambda \int_{\max(0,(t-1)/2)}^{t/2} e^{-\lambda u} du && (19) \\ &= \begin{cases} 0.5\lambda^2 t e^{-\lambda t} + 0.5 e^{-\lambda t/2}(e^{\lambda/2} - 1) & 1 \le t \\ 0.5\lambda^2 t e^{-\lambda t} + 0.5(1 - e^{-\lambda t/2}) & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} && (20) \end{aligned}$$

Notice that this pdf has a part which is proportional to an Erlang pdf ($te^{-\lambda t}$), and a part which is proportional to an exponential pdf with half the rate of the original process ($e^{-\lambda t/2}$). These correspond to the exponential part of $f_{U_2|U_1}(v|u)$, and the uniform part. One can imagine that the random variable U_2 selects from two equally probable options: either U_2 choose to be an exponential random variable independent of U_1 (in which case T_2 is Erlang), or U_2 chooses to take a value which is very close to the value of U_1 (in which case T_2 is just the exponential random variable for a Poisson process with rate equal to half the rate of the original process).

4. **[Quasi-Gaussian Random Variables]**

(a) $E[Y] = E[X_1] + E[X_2] + E[X_3] = 0.5 + 0.5 + 0.5 = 1.5$. $\text{Var}(Y) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = 0.25$.

(b) This problem is most conveniently solved by assuming that $S = X_1 + X_2$; then $Y = S + X_3$. The pdf $f_S(c)$ is already given in Fig. 4.16 in the notes. Performing the convolution $f_Y(v) = f_S(v) * f_X(v)$ yields

$$f_Y(v) = \begin{cases} \frac{1}{2}v^2 & 0 \le v \le 1 \\ \frac{3}{4} - \left(v - \frac{3}{2}\right)^2 & 1 \le v \le 2 \\ \frac{1}{2}(3-v)^2 & 2 \le v \le 3 \\ 0 & \text{otherwise} \end{cases}$$

The sketch should be piece-wise quadratic, with a concave quadratic between $(v, f_Y(v)) = (0,0)$ and $(v, f_Y(v)) = (1, 0.5)$, a convex quadratic between $v = 1$ and $(v, f_Y(v)) = (2, 0.5)$, and a concave quadratic between $v = 2$ and $(v, f_Y(v)) = (3, 0)$.

(c) i. $F_Y(0.1)$
$F_Y(0.1) = \int_0^{0.1} \frac{1}{2}v^2 dv = \frac{1}{60} \approx 0.0167$

ii. $F_{\tilde{Y}}(0.1)$
$F_{\tilde{Y}}(0.1) = \Phi\left(\frac{0.1-1.5}{\sqrt{0.25}}\right) = \Phi(-2.8) = 0.0026$

iii. $F_Y(1.4)$

$$F_Y(1.4) = \int_0^1 \frac{1}{2}v^2 dv + \int_1^{1.4} \left(\frac{3}{4} - (v - \frac{3}{2})^2\right) dv = 0.42533$$

iv. $F_{\tilde{Y}}(1.4)$
$F_{\tilde{Y}}(1.4) = \Phi\left(\frac{1.4-1.5}{\sqrt{0.25}}\right) = \Phi(-0.2) = 0.4207$