ECE 313: Problem Set 10: Solutions
Jointly distributed random variables including independent random variables

1. **[Joint probability mass functions]**

   (a) Given a fixed number of smartphones $S = n$, the number of iPhones, I, is a binomial random variable with parameters n and $1/2$. Also, $P\{I = i, A = j\} = P\{I = i, S = i + j\} = P\{I = i|S = i + j\}P\{S = i + j\}$. Putting these two facts together yields the numerical answer, given in the following table.

$i \downarrow \diagdown j \rightarrow$	0	1	2	3
0	0.15	0.15	0.0875	0.025
1	0.15	0.175	0.075	0.0
2	0.0875	0.075	0.0	0.0
3	0.025	0.0	0.0	0.0

   (b) By summing over rows in the table,

$$p_I(i) = P\{I = i\} = \begin{cases} 0.15 + 0.15 + 0.0875 + 0.025 = 0.4125 & i = 0 \\ 0.15 + 0.175 + 0.075 = 0.4 & i = 1 \\ 0.0875 + 0.075 = 0.1625 & i = 2 \\ 0.025 & i = 3 \end{cases}$$

   (c) By symmetry, I and A have the same marginals. Hence,

$$p_A(j) = \begin{cases} 0.4125 & j = 0 \\ 0.4 & j = 1 \\ 0.1625 & j = 2 \\ 0.025 & j = 3 \end{cases}$$

   (d) $P\{I \neq A\} = 1 - P\{I = A\} = 1 - (0.15 + 0.175) = 0.675.$

   (e) $P\{I = 1|A = 2\} = \frac{P\{I=1,A=2\}}{P\{A=2\}} = \frac{0.075}{0.1625} \approx 0.4615.$

2. **[Joint probability mass functions]**

   (a) There are five packets and two have errors, therefore, there are $\binom{5}{2} = 10$ possible orderings of the two errored packets among the five packets. Each one of these orderings specifies a unique pair (N_1, N_2), so the probability of any pair is $1/10$. Hence, $P\{N_1 = i, N_2 = j\} = \frac{1}{10}$ for $1 \le i \le 4$ and $1 \le j \le 5 - i$. It is zero else.

   (b) Each pair (N_1, N_2) with non-zero probability is equally likely, and for each $1 \le i \le 4$, $1 \le j \le 5 - i$. Hence $p_{N_1}(i) = P\{N_1 = i\} = \frac{1}{10}(5 - i)$ for $1 \le i \le 4$, and it is zero else.

   (c) Each pair (N_1, N_2) with non-zero probability is equally likely, and for each $1 \le j \le 4$, $1 \le i \le 5 - j$. Hence $p_{N_2}(j) = P\{N_2 = j\} = \frac{1}{10}(5 - j)$ for $1 \le j \le 4$, and it is zero else.

(d) They are not independent.
One reason is that $P\{N_1 = i, N_2 = j\} \neq P\{N_1 = i\}P\{N_2 = j\}$. For example, $P\{N_1 = 3, N_2 = 3\} = 0 \neq P\{N_1 = 3\}P\{N_2 = 3\}$.

(e) $P\{N_1 = N_2\} = \sum_{i=1}^{2} P\{N_1 = N_2 = i\} = 2\frac{1}{10} = \frac{1}{5}$.

(f) $P\{N_1 \leq N_2\} = \sum_{j=1}^{4} \sum_{i=1}^{\min(j,5-j)} \frac{1}{10} = 6\frac{1}{10} = \frac{3}{5}$.

(g) $P\{N_1 = 2|N_2 = 2\} = \frac{P\{N_1=2,N_2=2\}}{P\{N_2=2\}} = \frac{1/10}{3/10} = \frac{1}{3}$.

3. **[Joint probability density functions]**

(a) To have a valid joint density one needs

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dvdu = \int_{0}^{\infty}\int_{-u}^{u} c\left(u^2 - v^2\right)e^{-u}dvdu = 8c.$$

Hence, $c = \frac{1}{8}$.

(b)

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)dv = \int_{-u}^{u} \frac{1}{8}\left(u^2 - v^2\right)e^{-u}dv = \frac{1}{6}u^3e^{-u},$$

for $u > 0$, and it is zero else.

(c) For $v > 0$,

$$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)du = \int_{v}^{\infty} \frac{1}{8}\left(u^2 - v^2\right)e^{-u}dv = \frac{1}{4}e^{-v}(1+v).$$

By symmetry, for $v < 0$, $f_Y(v) = \frac{1}{4}e^{v}(1-v)$.
Hence, $f_Y(v) = \frac{1}{4}e^{-|v|}(1+|v|)$ for $-\infty < v < \infty$.

(d) They are not independent.
One reason is that the support of $f_{X,Y}$ is not a product set.

(e) $P\{X + Y \geq 0\} = P\{Y \geq -X\} = 1$.

4. **[Joint probability density functions]**

(a) For $v \in [0,1]$,

$$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)du = \int_{0}^{1-v} \frac{3}{2}du + \int_{1-v}^{1} \frac{1}{2}du = \frac{3-2v}{2}.$$

It is zero else.

(b) The region of interest lies completely inside the region where the joint pdf is 3/2, hence we can obtain the desired probability by calculating the area of the desired region and multiplying it by 3/2. The area can be obtained by considering an isosceles right triangle with equal sides of length 1/2, i.e. $area = \frac{(1/2)(1/2)}{2} = \frac{1}{8}$. Therefore, $P\{X+Y \leq 1/2\} = \frac{1}{8}\frac{3}{2} = \frac{3}{16}$.

(c) The region of interest lies completely inside the region where the joint pdf is 1/2, hence we can obtain the desired probability by calculating the area of the desired region and multiplying it by 1/2. The area can be obtained by considering the area of a square of length one and substracting from it the area of a quarter circle of radius one, i.e. $area = (1)^2 - \frac{\pi(1)^2}{4} = \frac{4-\pi}{4}$. Therefore, $P\{X^2 + Y^2 \geq 1\} = \frac{4-\pi}{4}\frac{1}{2} = \frac{4-\pi}{8}$.

5. **[Joint probability density functions]**

(a) X and Y are independent and Y is exponentially distributed with parameter $\lambda = 2$. Therefore, $f_Y(v) = 2e^{-2v}$ for $v > 0$, and it is zero else.

(b) X and Y are independent, so their joint pdf is given by the product of their marginals. Hence,

$$P\{Y \geq X\} = \int_0^1 \int_u^\infty f_{X,Y}(u,v) dv du = \int_0^1 \int_u^\infty (1) \left(2e^{-2v}\right) dv du = \frac{1 - e^{-2}}{2}.$$