University of Illinois **Spring 2012**

ECE 313: Problem Set 12
Moments of Jointly Distributed Random Variables, Minimum Mean Square Error Estimation

Due: Wednesday, April 25 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.8–4.9.

1. **[Correlation of Discrete-Type Random Variables]**
Suppose that X_i and Y_i are jointly distributed with the following pmf.

Y_1 \ X_1	-1	0	1
-1	1/9	1/9	1/9
0	1/9	1/9	1/9
1	1/9	1/9	1/9

Y_2 \ X_2	-1	0	1
-1	1/6	0	1/6
0	0	1/3	0
1	1/6	0	1/6

Y_3 \ X_3	-1	0	1
-1	0	0	1/3
0	0	1/3	0
1	1/3	0	0

(a) Find the correlation, covariance, and correlation coefficient of (X_i, Y_i) for each $i = 1, 2, 3$.

(b) Assume that X_i and Y_j are independent if $i \neq j$. Assume that X_1, X_2, are X_3 are pairwise uncorrelated, and Y_1, Y_2, and Y_3 are pairwise uncorrelated. Let $X = 3X_1 + 2X_2 + X_3$ and $Y = Y_1 + 2Y_2 + 3Y_3$. Find $Cov(X, Y)$ and $\rho_{X,Y}$.

2. **[Correlation of Continuous-Type Random Variables]**
Suppose X and Y are jointly distributed with the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}(1 - |u-1| - |v|), & \text{if } -1 \leq v \leq 1 \text{ and } |v| \leq u \leq 2 - |v| \\ 0, & \text{else} \end{cases}$$

(a) Sketch the support of $f_{X,Y}$, and describe the shape of $f_{X,Y}$ on the support.

(b) Find $Cov(X, Y)$ and $\rho_{X,Y}$.

(c) Are X and Y uncorrelated? Are X and Y independent?

(d) Suppose $\binom{W}{Z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \binom{X}{Y}$. Express $\rho_{W,Z}$ with respect to a, b, c, and d. (Hint: You may use the fact $Var(X) = Var(Y)$, which should be clear from your answer to part (a). You do not need to calculate $Var(X)$.

3. **[Correlation of Random Variables with a Symmetric Joint Pdf]**
Suppose X and Y have joint pdf $f_{X,Y}(u,v)$ which is symmetric with respect to line $u = u_0$ for some constant u_0. Show that X and Y are uncorrelated. (Hint: You may need (i) $Cov(X - u_0, Y) = Cov(X, Y)$ and (ii) $\int_{-\infty}^{\infty} ug(u)du = 0$ for any even function $g(u)$.)

4. **[A Portfolio Selection Problem]**
Suppose you are an investment fund manager with three financial instruments to invest your funds in for a one year period. Assume that, based on past performance, the returns on

investment for the instruments have the following means and standard deviations.

Instrument	Expected value after one year	Standard deviation of value after one year
Stock fund (S)	$\mu_S = 1.10$ i.e., 10% expected gain	σ_S=0.15
Bond fund (B)	$\mu_B = 1.00$ i.e., expected gain is zero	σ_B=0.15
T-bills (T)	$\mu_T = 1.02$ i.e. 2% gain	σ_T=0

(So $T \equiv 1.02$.) Also assume the correlation coefficient between the stocks and bonds is $\rho_{S,B} = -0.8$. Some fraction of the funds is to be invested in stocks, some fraction in bonds, and the rest in T-bills, and at the end of the year the return per unit of funds is R. There is no single optimal choice of what values to use for these fractions; there is a tradeoff between the mean, μ_R, (larger is better) and the standard deviation, σ_R (smaller is better). Plot your answers to the problems below using a horizontal axis for mean return ranging from 1.0 to 1.1, and a vertical axis for standard deviation ranging from 0 to 0.15. Label the points $P_S = (1.1, 0.15)$, $P_B = (1, 0.15)$ and $P_T = (1.02, 0)$ on the plot corresponding to the three possibilities for putting all the funds into one instrument.

(a) Let $R_\lambda = \lambda S + (1 - \lambda)T$, so R_λ is the random return resulting when a fraction λ of the funds is put into stocks and a fraction $1 - \lambda$ is put into T-bills. Determine and plot the set of $(\mu_{R_\lambda}, \sigma_{R_\lambda})$ pairs as λ ranges from zero to one.

(b) Let $R_\alpha = \alpha S + (1 - \alpha)B$, so R_α is the random return resulting when a fraction α of the funds is put into stocks and a fraction $1 - \alpha$ is put into bonds. Determine and plot the set of $(\mu_{R_\alpha}, \sigma_{R_\alpha})$ pairs as α ranges from zero to one. (Hint: Use the fact $\rho_{S,B} = -0.8$.).

(c) Combining parts (a) and (b), let $R_{\lambda,\alpha} = \lambda R_\alpha + (1 - \lambda)T$, so $R_{\lambda,\alpha}$ is the random return resulting when a fraction $1 - \lambda$ of the funds is invested in T-bills as in part (a), and a fraction λ of the funds is invested in the same mixture of stock and bond funds considered in part (b). For each $\alpha \in \{0.2,\ 0.4,\ 0.6,\ 0.8\}$, determine and plot the set of $(\mu_{R_{\lambda,\alpha}}, \sigma_{R_{\lambda,\alpha}})$ pairs as λ ranges from zero to one. (Hint: You may express your answers in terms of $(\mu_{R_\alpha}, \sigma_{R_\alpha})$ found in part (b).)

(d) As α and λ both vary over the interval $[0, 1]$, the corresponding point $(\mu_{R_{\lambda,\alpha}}, \sigma_{R_{\lambda,\alpha}})$ sweeps out the set of *achievable* (mean, standard deviation) pairs. An achievable pair $(\widetilde{\mu}_R, \widetilde{\sigma}_R)$ is said to be strictly better than an achievable pair (μ_R, σ_R) if either ($\mu_R < \widetilde{\mu}_R$ and $\sigma_R \geq \widetilde{\sigma}_R$) or ($\mu_R \leq \widetilde{\mu}_R$ and $\sigma_R > \widetilde{\sigma}_R$.). An achievable pair $(\widetilde{\mu}_R, \widetilde{\sigma}_R)$ is *undominated* if there is no other achievable pair strictly better than it. Identify the set of *undominated* achievable pairs.

5. **[Minimum Mean Square Error Estimation]**
Suppose X and Y have the following joint pdf:

$$f_{X,Y}(u, v) = \begin{cases} u + v & \text{if } u \in [0, 1], v \in [0, 1], \\ 0 & \text{else.} \end{cases}$$

(a) Find the unconstrained estimator $g^*(u_0)$ of Y for given observation $X = u_0$.

(b) Find the linear estimator $L^*(u_0)$ for observation $X = u_0$.

6. **[Minimum Mean Square Error Estimation]**
Suppose X and Y are distributed with the following pdf:

$$f_{X,Y}(u, v) = \begin{cases} e^{-v}, & 0 \leq u \leq v \\ 0, & \text{else.} \end{cases} \tag{1}$$

(a) Sketch the support of $f_{X,Y}$.

(b) Find the unconstrained estimator $g^*(u_0)$ of Y for observation $X = u_0$,

(c) Find the linear estimator $L^*(u_0)$ of Y for obervation $X = u_0$, and then compare $L^*(u_0)$ and $g^*(u_0)$ for u_0 in the support of f_X.