**University of Illinois** **Spring 2012**

# ECE 313: Problem Set 11
## Independent RVs; Sums of RVs

**Due:** Wednesday, April 18 at 4 p.m.
**Reading:** *ECE 313 Course Notes,* Sections 4.4–4.6.

1. **[Independent Random Variables]**
   Determine, for each of the following joint distributions, whether or not $X$ and $Y$ are independent random variables.

   (a)
   $$p_{X,Y}(j,k) = \frac{(n!)^2}{j!k!(n-j)!(n-k)!}p^{j+k}(1-p)^{2n-j-k}, \quad 0 \le j \le n,\ 0 \le k \le n$$

   (b)
   $$p_{X,Y}(j,k) = \frac{n!}{j!k!(n-j-k)!}p^{j+k}(1-2p)^{n-j-k}, \quad 0 \le j \le n,\ 0 \le k \le n-j$$

   (c)
   $$f_{X,Y}(u,v) = \frac{1}{2\pi}e^{-\frac{1}{2}(u^2+v^2)}, \qquad -\infty < u < \infty,\ -\infty < v < \infty$$

   (d)
   $$f_{X,Y}(u,v) = \frac{\lambda^2}{4}e^{-\lambda|u|}e^{-\lambda|v|}, \qquad -\infty < u < \infty,\ -\infty < v < \infty$$

   (e)
   $$f_{X,Y}(u,v) = \frac{\lambda^2}{2}e^{-\lambda|u+v|}e^{-\lambda|u-v|}, \qquad -\infty < u < \infty,\ -\infty < v < \infty$$

   (f)
   $$f_{X,Y}(u,v) = \begin{cases} \frac{1}{12} & 1 \le u \le 4,\ 1 \le v \le 2 \\ \frac{1}{36} & 1 \le u \le 4,\ 3 \le v \le 6 \\ \frac{1}{4} & 5 \le u \le 6,\ 1 \le v \le 2 \\ \frac{1}{12} & 5 \le u \le 6,\ 3 \le v \le 6 \\ 0 & \text{otherwise} \end{cases}$$

2. **[Conditional Distributions]**
   For each of the distributions in problem 1, find the conditional distribution $p_{Y|X}(k|j)$ or $f_{Y|X}(v|u)$. You may find some of the summations in appendix 6.2 useful.

   (a)
   $$p_{X,Y}(j,k) = \frac{(n!)^2}{j!k!(n-j)!(n-k)!}p^{j+k}(1-p)^{2n-j-k}, \quad 0 \le j \le n,\ 0 \le k \le n$$

(b)

$$p_{X,Y}(j,k) = \frac{(n!}{j!k!(n-j-k)!}p^{j+k}(1-2p)^{n-j-k}, \quad 0 \le j \le n, \ 0 \le k \le n-j$$

(c)

$$f_{X,Y}(u,v) = \frac{1}{2\pi}e^{-\frac{1}{2}(u^2+v^2)}, \qquad -\infty < u < \infty, \ -\infty < v < \infty$$

(d)

$$f_{X,Y}(u,v) = \frac{\lambda^2}{4}e^{-\lambda|u|}e^{-\lambda|v|}, \qquad -\infty < u < \infty, \ -\infty < v < \infty$$

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$$f_{X,Y}(u,v) = \frac{\lambda^2}{2}e^{-\lambda|u+v|}e^{-\lambda|u-v|}, \qquad -\infty < u < \infty, \ -\infty < v < \infty$$

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$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{12} & 1 \le u \le 4, \ 1 \le v \le 2 \\ \frac{1}{36} & 1 \le u \le 4, \ 3 \le v \le 6 \\ \frac{1}{4} & 5 \le u \le 6, \ 1 \le v \le 2 \\ \frac{1}{12} & 5 \le u \le 6, \ 3 \le v \le 6 \end{cases}$$

3. **[Poisson Processes and Their Kin]**

(a) Roughly speaking, we can model the electrical voltage spikes on a neuron as the counts in a Poisson process, with some non-negative count rate $\lambda$. As you remember, the intercount times in a Poisson process are random variables with the following pdf:

$$f_U(u) = \begin{cases} \lambda e^{-\lambda u} & u \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let $T_2$ be the second-order count time of a Poisson process with rate $\lambda$. Recall from section 3.5.2 that $T_2$ can be written as the sum of two random variables, $T_2 = U_1 + U_2$, where $U_1$ and $U_2$ are independent and identically distributed (i.i.d.) random variables with pdf as given by Eq. 1. Express the relationship of $U_1$, $U_2$, and $T_2$ as a convolution by integrating over $u$. Show that solving the convolution integral yields an Erlang pdf for $f_U(u)$.

(b) A Poisson process is memoryless, but real-world processes often have some type of memory. For example, real-world neurons are nearly Poisson, except that after the neuron fires, there is a small "refractory period" of $b$ milliseconds during which it is unable to fire again. If we assume the neuron fires at time zero, so $U_1$ is the first time it fires after time zero and $U_1 + U_2$ is the second time it fires after time zero, then it is reasonable to assume that $U_1$ and $U_2$ are two first-order count times with pdfs $f_U(u)$ given by:

$$f_U(u) = \begin{cases} \lambda e^{-\lambda(u-b)} & u \ge b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Define $T_2 = U_1 + U_2$ to be the second-order count time in this process. Find the pdf $f_{T_2}(t)$.

(c) Equation 2 is a pretty good model of a neuron acting without external stimulation, but it is a pretty bad model of a neuron acting in the presence of external stimulation. For example, in response to external stimulation, many neurons will fire with a quasi-periodic firing rate, meaning that $U_2$ and $U_1$ are not independent: they are not perfectly identical, but they have a high probability of being approximately the same. A simple model of quasi-periodicity represents $f_{U_1,U_2}(u,v) = f_{U_1}(u) f_{U_2|U_1}(v|u)$, where $f_{U_1}(u)$ and $f_{U_2|U_1}(v|u)$ are defined as:

$$f_{U_1}(u) = \begin{cases} \lambda e^{-\lambda u} & u \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$f_{U_2|U_1}(v|u) = \begin{cases} 0.5 + 0.5\lambda e^{-\lambda v} & u \leq v < u+1 \\ 0.5\lambda e^{-\lambda v} & 0 \leq v < u \;\text{ or }\; u+1 \leq v \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Define $T_2 = U_1 + U_2$ to be the second-order count time in this process. Find $f_{T_2}(t)$.

4. **[Quasi-Gaussian Random Variables]**
Suppose that $X_1$, $X_2$ and $X_3$ are independent continuous random variables, each drawn from the following pdf:

$$f_X(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

And suppose that $Y = X_1 + X_2 + X_3$.

(a) Find $E[Y]$ and $\text{Var}(Y)$. Since $X_1$, $X_2$, and $X_3$ are independent, you may assume that $\text{Var}(Y) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3)$.

(b) Find $f_Y(v)$. Sketch $f_Y(v)$, showing the values at $v = 0$, $v = 1$, $v = 2$, and $v = 3$.

(c) Define $\tilde{Y}$ to be a Gaussian random variable with the same mean and variance as $Y$. Evaluate the Gaussian approximation for $Y$ near its edge, and near its center, by finding the following particular values:

i. $F_Y(0.1)$

ii. $F_{\tilde{Y}}(0.1)$

iii. $F_Y(1.4)$

iv. $F_{\tilde{Y}}(1.4)$