University of Illinois **Spring 2012**

ECE 313: Problem Set 9
Functions of a random variable, failure rate functions, and binary hypothesis testing for continuous-type observations

Due:	Wednesday March 28 at 4 p.m.
Reading:	*ECE 313 Notes* Sections 3.8-3.10.
Note:	The final exam has been scheduled for Monday, May 7, 1:30-4:30 pm

1. [**A binary quantizer with Laplacian input**]
Suppose X has pdf $f_X(u) = \frac{e^{-|u|}}{2}$ and $Y = g(X)$ where $g(u) = \alpha(\text{sign}(u)) = \begin{cases} \alpha & \text{if } u \geq 0 \\ -\alpha & \text{if } u < 0 \end{cases}$ for some constant α. So Y is the output of a binary quantizer with input X.
   (a) Describe the pdf or pmf of Y.
   (b) Find the mean square quantization error, $E[(X-Y)^2]$. Your answer should depend on α. (Hint: $\int_0^\infty u^k e^{-u} du = k!$ for nonnegative integers k.)
   (c) Find α to minimize the mean square quantization error.

2. [**Function of a random variable**]
Let X have pdf $f_X(u) = \frac{1}{2u^2}$ for $|u| \geq 1$ and $f_X(u) = 0$ for $|u| < 1$. Let $Y = \sqrt{|X|}$.
   (a) Using LOTUS, find $E[Y]$.
   (b) Find the pdf of Y.
   (c) (This part does not involve Y.) Find the nondecreasing function h so that $h(X)$ is uniformly distributed over $[0, 1]$. Be as explicit as possible.

3. [**Linearization of a quadratic function of a random variable**]
Suppose $Y = g(X)$ where $g(u) = 8u^2$ and X is uniformly distributed over $[9.9, 10.1]$. (For example, Y could be the total energy stored in a capacitor if X is the voltage across the capacitor.)
   (a) Using LOTUS and the fact $\text{Var}(Y) = E[Y^2] - E[Y]^2$, find the mean and variance of Y.
   (b) Find and sketch the pdf of Y.
   (c) Note that X is always relatively close to 10. The first order Taylor approximation yields that $g(u) \approx g(10) + g'(10)(u - 10) = 800 + 160(u - 10)$ for u near 10. Let $Z = 800 + 160(X - 10)$. We expect Z to be a good approximation to Y. Identify the probability distribution of Z.
   (d) Find the mean and variance of Z.
   (e) Your answers to (a)-(d) should show that Y and Z have nearly the same pdfs, means, and variances. To get another idea of how close Y and Z are, compute $E[(Y - Z)^2]$. (Hint: Express $(Y - Z)^2$ as a simple function of X and use LOTUS.)

4. [**Generation of a random variable with a given failure rate function**]
Suppose $(r(t), t \geq 0)$ is a positive, continuous function with $\int_0^\infty r(t)dt = \infty$. Let X be an exponentially distributed random variable with parameter one. Let T be implicitly determined by X through the equation $\int_0^T r(s)ds = X$. For example, if X is the amount of water in a well at time zero, and if water is scheduled to be drawn out with a time-varying flow rate $r(t)$, then the well becomes dry at time T.
   (a) Express the CDF of T in terms of the function r. (Hint: For any $t \geq 0$, the event $\{T \leq t\}$ is equivalent to $\left\{X \leq \int_0^t r(s)ds\right\}$. Using the analogy above, it is because the well is dry at time t if and only if X is less than or equal to the amount of water scheduled to be drawn out by time t.
   (b) Express the failure rate function h of T in terms of the function r.

5. **[A simple hypothesis testing problem]**
On the basis of a sensor output X, it is to be decided which hypothesis is true: H_0 or H_1. Suppose that if H_0 is true then X has density f_0 and if H_1 is true then X has density f_1, where the densities are given by

$$f_0(u) = \begin{cases} \frac{1}{2} & |u| \leq 1 \\ 0 & |u| > 1 \end{cases} \qquad f_1(u) = \begin{cases} |u| & |u| \leq 1 \\ 0 & |u| > 1 \end{cases}$$

   (a) Describe the ML decision rule for deciding which hypothesis is true for observation X.
   (b) Find $p_{false\ alarm}$ and p_{miss} for the ML rule.
   (c) Suppose it is assumed *a priori* that H_0 is true with probability π_0 and H_1 is true with probability $\pi_1 = 1 - \pi_0$. For what values of π_0 does the MAP decision rule declare H_1 with probability one, no matter which hypothesis is really true?
   (d) Suppose it is assumed *a priori* that H_0 is true with probability π_0 and H_1 is true with probability $\pi_1 = 1 - \pi_0$. For what values of π_0 does the MAP decision rule declare H_0 with probability one, no matter which hypothesis is really true?

6. **[(COMPUTER EXERCISE) Running averages of independent, identically distributed random variables]**
Consider the following experiment, for an integer $N \geq 1$. (a) Suppose $U_1, U_2, \ldots, U_N$ are mutually independent, uniformly distributed random variables on the interval $[-0.5, 0.5]$ Let $S_n = U_1 + \cdots + U_n$ denote the cumulative sum for $1 \leq n \leq N$. Simulate this experiment on a computer and make two plots, the first showing $\frac{S_n}{n}$ for $1 \leq n \leq 100$ and the second showing $\frac{S_n}{n}$ for $1 \leq n \leq 10000$. (b) Repeat part (a), but change S_n to $S_n = Y_1 + \cdots + Y_n$ where $Y_k = \tan(\pi U_k)$. (This choice makes each Y_k have the Cauchy distribution, $f_Y(v) = \frac{1}{\pi(1+v^2)}$; see Example 3.8.5 of the notes. Since the pdf f_Y is symmetric about zero, it is tempting to think that $E[Y_k] = 0$, but that is *false*; $E[Y_k]$ is not well defined because $\int_0^\infty v f_Y(v)dv = +\infty$ and $\int_{-\infty}^0 v f_Y(v)dv = -\infty$. Thus, for any function $g(n)$ defined for $n \geq 1$, it is possible to select $a_n \to -\infty$ and $b_n \to \infty$ so that $\int_{a_n}^{b_n} v f_Y(v)dv = g(n)$ for all n. It is said, therefore, that the integral $\int_{-\infty}^{\infty} v f_Y(v)dv$ is *indeterminate.)*