University of Illinois **Spring 2012**

ECE 313: Problem Set 7
Continuous RVs and Poisson Processes

Due: Wednesday, March 7 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.2–3.5.

1. **[Power-Law Distributions]**
A power-law distribution is a pdf of the form

$$f_X(u) = \begin{cases} Au^{-b} & u \geq 1 \\ 0 & \text{else} \end{cases}$$

Few physical phenomena follow power-law distributions, but many social phenomena are distributed in this way. For example, if you choose an individual uniformly from the set of all humans on Earth, his or her annual income is approximately a power-law random variable. Discretized power-law distributions are also good models of the number of books sold by any given author, the number of times any given word is used in any given TV broadcast, and the number of living speakers of any given language. Power-law distributions are unusual in many respects. First, $f_X(u)$ is not a valid pdf for all possible values of the parameter b. Second, there are some values of b for which $f_X(u)$ is a valid pdf, but for which the expected value of X is unbounded (you can write "$E[X] = +\infty$"). Third, there are some values of b for which $E[X]$ is a well-behaved finite number, but for which $\mathrm{Var}(X) = +\infty$. The point of this problem is to explore some of these cases.

   (a) For each of the following values of the parameter b, either (1) find the value of A such that $f_X(u)$ is a valid pdf, or (2) prove that no such value exists.
      i. $b = 2$.
      ii. $b = 1.1$.
      iii. $b = 0.5$.
      iv. $b = 1$.

   (b) Find the set of parameter pairs (A, b) for which $f_X(u)$ is a valid pdf.

   (c) Suppose $b = 3$. What is $E[X]$?

   (d) Suppose $b = 2$. What is $E[X]$?

   (e) Suppose $b = 3$. What is $\mathrm{Var}(X)$?

2. **[Uniform Quantization Errors]**
Suppose that you have recorded an audio signal from a microphone, and that you have normalized the audio signal so that its samples, X, are continuous random variables, uniformly distributed over the range $-1 \leq X \leq 1$. In order to convert an analog audio signal into a digital audio signal, the most common strategy is uniform quantization. The real-valued signal X is encoded on hard disk using a binary integer C, computed according to

$$C = \begin{cases} 0 & X < -1 \\ \lfloor 2^{B-1}(X+1) \rfloor & -1 \leq X < 1 \\ 2^B - 1 & X \geq 1 \end{cases} \tag{1}$$

where B is a positive integer-valued parameter called the bit rate (measured in bits per audio sample), and $c = \lfloor y \rfloor$ is defined to be the largest integer such that $c \leq y$. Notice that, by this definition, C is a discrete random variable defined over the range $C \in \{0, \ldots, 2^B - 1\}$.

Equation (1) is an information-losing transformation: once the audio signal has been digitized, it is not possible to reconstruct the original signal without error. The best we can do, usually, is to approximate the original signal using the approximate value $\hat{X}$, defined as:

$$\hat{X} = 2^{-(B-1)}C - (1 - 2^{-B}) \tag{2}$$

The quantization error, Q, is defined to be the difference between the reconstructed and original signals, thus

$$Q \equiv \hat{X} - X \tag{3}$$

(a) What is $E[X]$?

(b) What is $E[X^2]$? This quantity is often called the "power" of the signal X.

(c) Suppose $B = 2$ bits/sample.

i. Sketch $f_X(u)$, the pdf of X. Shade the region under the portion of this pdf corresponding to the event "$C = 3$." Based entirely on your sketch, and without doing any explicit integrals, find $P\{C = 3\}$.

ii. Sketch $f_X(u)$, the pdf of X. Shade the region under the portion of this pdf corresponding to the event "$|\hat{X}| = \frac{1}{4}$" (in words: the signal X is quantized using some codeword C, then reconstructed using a value $\hat{X}$ such that the absolute value of $\hat{X}$ is $\frac{1}{4}$). Based entirely on your sketch, and without doing any explicit integrals, find $P(|\hat{X}| = \frac{1}{4})$.

(d) Now allow B to be a free parameter. Let Q be the continuous random variable $Q = \hat{X} - X$, and define its pdf to be $f_Q(v)$. Sketch $f_Q(v)$, and label both axes in terms of B.

(e) The signal-to-noise ratio level (SNRL) is defined to be

$$\text{SNRL} = 10 \log_{10} \left(\frac{E[X^2]}{E[Q^2]} \right)$$

Find the SNRL in terms of B.

3. **[The Laplacian: A Symmetrized Exponential]**
Samples of a speech or music signal are well modeled by a Laplacian pdf, defined as

$$f_X(u) = Ae^{-\lambda|u|}, \quad -\infty < u < \infty$$

where $\lambda > 0$.

(a) What is the value of A, in terms of λ?

(b) Find the CDF, $F_X(c)$, by integrating $f_X(u)$.

(c) What is $P\{|X| \leq c\}$, as a function of c?

(d) Many interesting facts about X can be most quickly derived by using the law of total probability, combined with the following trick. Define an exponential random variable Y,

$$f_Y(v) = \begin{cases} \lambda e^{-\lambda v} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then define the following experiment: First, Y is chosen at random, according to its pdf. Second, a fair coin is flipped. If the coin shows heads (event H), then X is set to $X = Y$. If the coin shows tails (event $T = H^c$), then X is set to $X = -Y$. Note that event H is independent of the value of random variable Y. By the law of total probability, this experiment implies the following formula; prove that this formula is satisfied by the CDF you found in part (b):

$$P(X \leq c) = P(H)P\{X \leq c|H\} + P(T)P\{X \leq c|T\} = P(H)P\{Y \leq c\} + P(T)P\{Y \geq -c\}$$

Be sure that your answer is correct regardless of whether $c \leq 0$ or $c \geq 0$.

(e) Find $E[X]$ using the law of total probability.

(f) Find $E[X^2]$ using the law of total probability.

(g) Find $\mathrm{Var}(X)$.

4. **[Wilma the Wombat]**
The video game "Wilma the Wombat" depicts the adventures of Wilma, a wombat in the Australian Outback, who is trying to cross a highway without getting hit by a car. The time, in minutes, between any two successive cars, T, is an exponential random variable,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

It takes Wilma exactly six minutes to cross the road. If a car comes during that time, Wilma bounces back to her own side of the road, and must begin again.

(a) What is the probability that Wilma successfully crosses the road on her first attempt?

(b) Wilma decides to try the following strategy. Instead of starting across the road immediately, she waits for the first car. Immediately after the first car passes, she starts across the road. Using this strategy, what is the probability that she successfully crosses the road on her first attempt?

(c) Being a talented hacker, you decide to change the rules of the game. You alter the code so that, instead of being hit by each car that passes, Wilma just ducks under the cars. A counter keeps track of the number of cars that pass, N, during the six minutes of Wilma's journey. What is the pmf, $p_N(i)$, of random variable N?

(d) Continuing your modification of the game, you decide to add a magic gumdrop on the center line of the highway. Suppose it takes Wilma 3 minutes to reach the magic gumdrop, then she eats it instantaneously, then immediately begins her trek across the remainder of the road. During the remaining 3 minutes of her trek across the highway, if a car arrives, she eats the car, gaining extra points. Her total score for the game is as follows:

$$X = \begin{cases} 0 & \text{if any car arrives in } 0 \leq t < 3 \\ N & \text{if no cars arrive in } 0 \leq t < 3, \text{ and } N \text{ cars arrive in } 3 \leq t < 6 \end{cases}$$

What is the pmf, $p_X(k)$, of random variable X?

(e) Suppose, now, that Wilma has been given a sack containing an infinite number of magic gumdrops, but that each one lasts for only one minute. In order to win the game, Wilma must eat three cars. In this new version of the game, Wilma is allowed to stay on the road as long as she likes, but in order to remain invulnerable (and continue eating cars), she must keep eating gumdrops at a rate of one per minute.

i. Define S to be the time, in minutes, until arrival of the third car. What is the pdf, $f_S(t)$, of random variable S?

ii. Define G to be the number of gumdrops that Wilma eats before arrival of the first car (assuming that she is eating gumdrops at a rate of one per minute). Find the pmf $p_G(g)$.

iii. Define D to be the number of gumdrops that Wilma eats, from the time she begins the game until arrival of the third car (assuming that she is eating gumdrops at a rate of one per minute). Find the pmf $p_D(d)$.