University of Illinois Spring 2012

ECE 313, Section F: Hour Exam I

Monday March 5, 2012
7:00 p.m. — 8:15 p.m.
163 Everitt

1. **[25 points]** The average temperature in Urbana in March, X, is a random variable distributed according to the following pmf:

$$p_X(k) = \begin{cases} \frac{1}{20}\,|k-10| & 10-b \le k \le 10+b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   (a) **[6 points]** What is b?
   Solution: $1 = \sum_{k=10-b}^{10+b} p_X(k) = \frac{2}{20}\sum_{n=1}^{b} n = \frac{b(b+1)}{20}$, so $b = 4$.

   (b) **[7 points]** The monthly cost of heating Everitt Lab, in dollars, is $Y = 1000(25-X)^2$. You may assume that $E[X] = 10$ and $\text{Var}(X) = 10$. What is $E[Y]$?
   Solution: $E[Y] = 625,000 - 50,000E[X] + 1000E[X^2]$. Since the pmf is symmetric, $E[X] = 10$. $E[X^2] = \text{Var}(X) + E^2[X] = 10 + (10)^2 = 110$. Therefore $E[Y] = 235,000$.

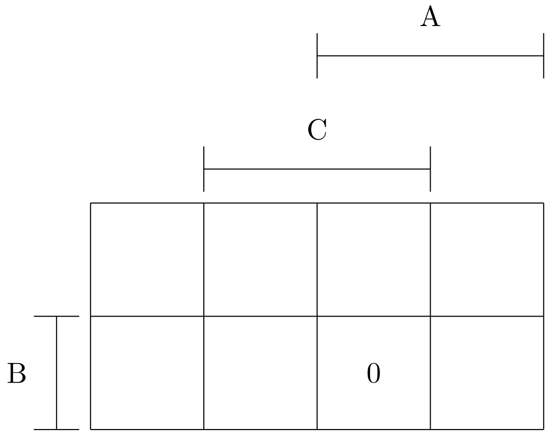
   (c) **[6 points]** Your cell phone reports the average temperature only in five-degree increments, but you are writing a report for the student senate that requires you to guess the temperature exactly. What is $P\{10 \le X \le 12\,|10 \le X \le 14\}$?
   Solution:

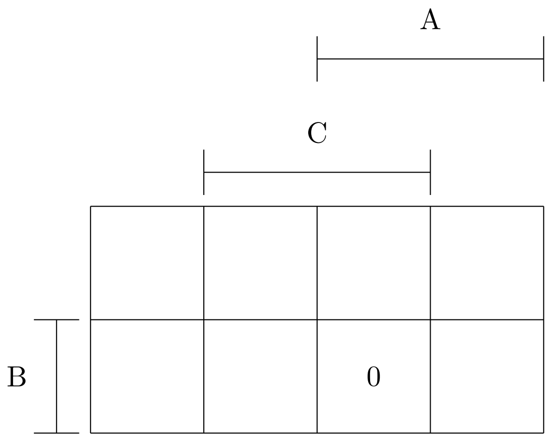
$$P\{10 \le X \le 12\,|10 \le X \le 15\} = \frac{P\{10 \le X \le 12\}}{P\{10 \le X \le 15\}} = \frac{3/20}{10/20} = 0.3$$

   (d) **[6 points]** Define the following events:
   - A: The average temperature is $X \le 10$.
   - B: There are at least 12 cloudy days in the month.

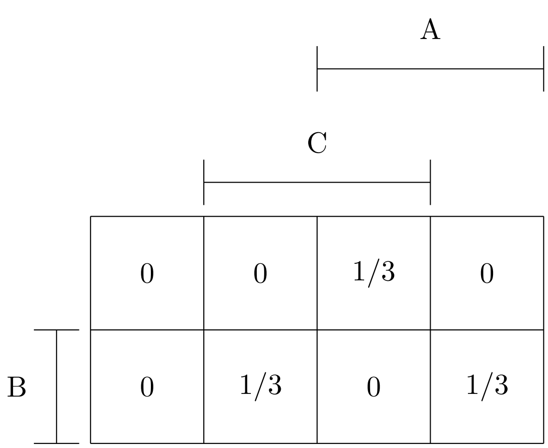
   A and B are independent events, $P(B) = 0.3$, and $P(A)$ is as specified in Eq. 1. What is $P(A^cB^c)$?
   Solution: The pmf is symmetric, so $P(A) = 0.5$. Therefore $P(A^c) = 0.5$, $P(B^c) = 0.7$, and $P(A^cB^c) = 0.35$.

2. **[10 points]** Suppose A, B, and C are events for a probability experiment such that $P(A) = P(B) = P(C) = 2/3$, and $P(ABC) = 0$. Fill in the probabilities of all events indicated in the Karnaugh map provided. Show your work. An extra copy is provided for you to show your final answer. Circle your final answer.





Solution: $P(A) = P(ABC^c) + P(AB^c) = 2/3$. Since $P(C^c) = 1/3$, it must be true that $P(ABC^c) \leq 1/3$; since $P(B^c) = 1/3$, it must be true that $P(AB^c) \leq 1/3$. The solution therefore requires that $P(ABC^c) = 1/3$ and $P(AB^c) = 1/3$. Similar computations give $P(ABC^c) = P(AB^cC) = P(A^cBC) = 1/3$, and the probability of any other event is identically zero.



A more tedious but more foolproof method for finding the solution is as follows. Let a, b, c, d, e, f, g be the variables needed to fill the seven blank spaces in the Karnaugh map. The following seven equations are true:

$$\begin{aligned}
1 &= a+b+c+d+e+f+g \\
1/3 &= a+b+c+d \\
2/3 &= e+f+g \\
1/3 &= a+e+d+g \\
2/3 &= b+c+f \\
1/3 &= a+b+e+f \\
2/3 &= c+d+g
\end{aligned}$$

Systematic reduction of these seven equations (e.g., begin by using the three shortest equations to find that $e = b + 2c + d$, $f = 2/3 - b - c$, $g = 2/3 - c - d$) leads to the correct answer.

3. **[15 points]** A particular sandwich bar offers customers their choice of ingredients. Every customer must choose exactly three different ingredients for his or her sandwich. There are five ingredients available: {Avocado, Broccoli, Cheese, Duck, and Egg}. If two sandwiches contain exactly the same ingredients, we say that they are the "same sandwich;" if at least one ingredient differentiates two sandwiches, we say that they are "different sandwiches."

   (a) **[5 points]** How many different sandwiches can be made that include Duck as one of the ingredients?

   Solution: There are $\binom{4}{2} = 6$ ways to choose the remaining two ingredients.

   (b) **[5 points]** Three customers in a row choose sandwiches. Each customer chooses three different ingredients uniformly at random from the set of five possible ingredients. What is the probability that two customers choose the same sandwich, but the third customer chooses a different sandwich?

   Solution: This probability is equal to a fraction. The denominator of the fraction is the number of different ways in which one can make three sandwiches, which is $\binom{5}{3}^3 = 1000$. The numerator is equal to the number of different ways in which one can make three sandwiches, of which two are the same, and one is different. There are 3 ways to choose the sandwich that will be different from the others. Once it has been chosen, it can be any one of $\binom{5}{3} = 10$ different sandwiches, and the other two can be any of the remaining $\binom{5}{3} - 1 = 9$ sandwiches. The probability that two customers have the same sandwich but the third sandwich differs is therefore $270/1000 = 27/100$.

   (c) **[5 points]** Alice's sandwich is different from Bob's (at least one of the three ingredients on Alice's sandwich is not found on Bob's sandwich). What's the probability that these sandwiches have exactly two ingredients in common?

   Solution: There are $\left(\binom{5}{3} - 1\right) = 9$ ways in which Bob can make his sandwich differ from Alice's, each of which includes one ingredient different from Alice's sandwich. The number of sandwiches containing two ingredients from Alice's sandwich, and one of the other ingredients, is $\binom{3}{2}\binom{2}{1} = 6$, so the probability that they share one ingredient is $6/9 = 2/3$.

4. **[25 points]** The t^{th} day is a good day with probability $P(G_t) = p$; otherwise it is a bad day. Events G_t and G_s are independent for $t \neq s$.

   (a) **[6 points]** Let X be the number of good days in any particular week. Suppose that $p = 0.5$. What is $P\{X \geq 2\}$?

   Solution:

$$P\{X \geq 2\} = 1 - p_X(0) - p_X(1) = 1 - (1-p)^7 - 7p(1-p)^6 = 1 - \frac{1}{128} - \frac{7}{128} = \frac{15}{16}$$

(b) **[6 points]** Let D be the number of bad days that occur in a row, prior to the first good day (notice that $D = 0$ is a possibility, therefore D is not a geometric random variable). Suppose that $p = 0.5$. What is $E[D^2]$?

Solution: D is not a geometric random variable, but $D = F - 1$ for geometric random variable F. $E[D] = E[F] - 1 = 1/p - 1 = 1$, and $\text{Var}(D) = \text{Var}(F) = p/(1-p)^2 = 2$, so $E[D^2] = \text{Var}(D) + E^2[D] = 3$.

(c) **[6 points]** Let Y be the number of good days that occur in some particular 100-day period, and assume that $p = 0.5$. Use the Chebyshev bound to find a lower bound on the quantity $P\{41 < Y < 59\}$.

Solution: The Chebyshev bound is

$$P\{|Y - E[Y]| \geq d\} \leq \frac{\sigma^2}{d^2} \tag{2}$$

$$P\{|Y - 50| \geq 9\} \leq \frac{100/4}{81} \tag{3}$$

$$P\{|Y - 50| < 9\} \geq \frac{81 - 25}{81} = \frac{56}{81} \tag{4}$$

(d) **[7 points]** Let Z be the number of good days that occur in some particular n-day period. For this part of the problem, assume that p is an unknown small number, and n is an unknown large number, but assume that $E[Z] = 6$. Use the Poisson approximation to find $P\{Z \leq 2\}$.

Solution:

$$P\{Z \leq 2\} = p_Z(0) + p_Z(1) + p_Z(2) \tag{5}$$

$$\approx e^{-5}\left(1 + 6 + \frac{6^2}{2!}\right) \tag{6}$$

$$= 25e^{-5} \tag{7}$$

5. **[25 points]** Let X be the number of microphones carried by a spy. The number of microphones carried by a spy from Organization A is a random variable with the following pmf:

$$p_A(k) = \begin{cases} \frac{1}{55}(10-k) & 0 \leq k \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

The number of microphones carried by a spy from Organization B is a random variable with the following pmf:

$$p_B(k) = \begin{cases} \frac{k}{55} & 1 \leq k \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Any given spy is from Organization A with probability $\pi_A = 0.8$, otherwise he or she is from Organization B.

(a) **[6 points]** You meet a spy. What is the probability that he or she is carrying no microphones?

Solution: $p_X(0) = \pi_A p_A(0) + \pi_B p_B(0) = 0.8\frac{10}{55} + 0 = \frac{8}{55}$

(b) **[7 points]** You discover that the spy is carrying at least one microphone ($X \geq 1$). Given only this one piece of information, what is the probability that this spy is from Organization B?

Solution:

$$P(B|X \geq 1) = \frac{\pi_B(1 - p_B(0))}{\pi_A(1 - p_A(0)) + \pi_B(1 - p_B(0))} = \frac{0.2}{0.8(45/55) + 0.2} = \frac{11}{47}$$

(c) Your mission is to find spies, to count the number of microphones each spy is carrying, and to determine, with no other information available to you, to which organization each spy belongs (Organization A or Organization B). Specify the ML and MAP decision rules for this problem; break ties in favor of Organization B. In both cases, you should be able to specify a rule of the form "Choose Organization B if $X \geq d$, otherwise choose Organization A," for some particular value of d.

i. **[6 points]** ML Decision Rule:
Solution: The ML decision rule is to choose B if $\frac{k}{55} \geq \frac{(10-k)}{55}$, i.e., if $k > 5$.

ii. **[6 points]** MAP Decision Rule:
Solution: The MAP decision rule is to choose B if $0.2\frac{k}{55} \geq 0.8\frac{(10-k)}{55}$, i.e., if $k \geq 8$.