University of Illinois Spring 2012

ECE 313: Conflict Final Exam

Tuesday, May 8, 2012
1:30 p.m. — 4:30 p.m.
165 Everitt Laboratory

1. **[25 points]** Farmland in the Midwest is divided into sections, each of which is one square mile. The owner of one section wants to divide it into three rectangular fields, of lengths X, Y, and Z, such that $X + Y + Z = 1$.

   (a) **[5 points]** Assume, for this part of the problem only, that X and Y are uniformly distributed over the support $\{(X, Y) = (u, v) : u \geq 0, v \geq 0, u + v \leq 1\}$. What is $E[Z]$?

   (b) **[10 points]** Assume, for this part of the problem only, that X is uniformly distributed between 0 and 1, and that Y is conditionally distributed as follows:

   $$f_{Y|X}(v|u) = \begin{cases} \frac{1}{1-u} & 0 \leq v < 1 - u \\ 0 & \text{otherwise} \end{cases}$$

   for $0 < u \leq 1$. Define $R = X + Y$. Find $f_R(r)$. Be sure to specify its value for all possible values of r, not just the nonzero values.

   (c) **[10 points]** Assume, for this part of the problem only, that the farmer is able to buy land from his neighbor as necessary so that X and Y are independent random variables, each of which is uniformly distributed between 0 and 1. Define $Z = 1 - X - Y$. Notice that, with this definition, it is possible for Z to be negative. Sketch $f_Z(w)$. Be sure to label the axes, in order to show the maximum value of $f_Z(w)$, and the values of w at all turning points in the function (all points at which the slope changes).

2. **[20 points]** Suppose Bob plays a series of card games at a casino. He declares prior to each game how much money he bets on the game. If he wins the game, he wins as much money as he bet. Otherwise, he loses as much money as he bet. Assume that he wins each game with probability 0.5, and the outcomes of the games are mutually independent. Suppose he initially has \$1,048,576 ($= 2^{20}$) and, for each game, he bets one half of the money he has prior to the game. (Assume that his bet can be any positive real, i.e., \$2/3.)

   (a) **[10 points]** Let X be the number of wins in the first 196 games, and let S be the money remaining after 196 games. Express S in terms of X. Simplify your answer as much as possible.

   (b) **[10 points]** Using the Gaussian approximation, express the probability Bob has \$1 or less after 196 games in terms of the Q-function or Φ-function. (Hint: Approximate $\log_2(3)$ by 1.6.)

3. **[20 points]** Consider a binary hypothesis testing problem with observation X. Under H_0, X has the binomial distribution with parameters $n = 72$ and $p = \frac{1}{3}$. Under H_1, X has the binomial distribution with parameters $n = 72$ and $p = \frac{2}{3}$.

   (a) **[5 points]** Describe the maximum likelihood decision rule for an observation k, where k is an arbitrary integer with $0 \leq k \leq 72$. Express the rule in terms of k as simply as possible. To be definite, in case of a tie in likelihoods, declare H_1 to be the hypothesis.

   (b) **[5 points]** Suppose a particular decision rule declares that H_1 is the true hypothesis if and only if $X \geq 34$. Express the approximate value of $p_{false\ alarm}$ for this rule in terms of the Q function, where $Q(c) = \int_c^\infty \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)du$. (To be definite, don't use the continuity correction.)

(c) **[5 points]** Describe the MAP decision rule for an observation k, where k is an arbitrary integer with $0 \le k \le 72$, for the prior distribution $\pi_0 = 0.9$ and $\pi_1 = 0.1$. Express the rule in terms of k as simply as possible.

(d) **[5 points]** Assuming the same prior distribution as in part (c), find $P(H_0|X = 38)$.

4. **[20 points]** Suppose X has the exponential distribution with parameter $\lambda > 0$. Express the answers below in terms of λ.

   (a) **[5 points]** Find $E[X^2]$.

   (b) **[7 points]** Find $P\{\lfloor X^2 \rfloor = 3\}$, where $\lfloor v \rfloor$ is the greatest integer less than or equal to v.

   (c) **[8 points]** Find the cumulative distribution function (CDF) of $Y = e^{-X}$. Be sure to specify it over the entire real line.

5. **[10 points]** Suppose $X_1, \ldots X_n$ and $Y_1, \ldots, Y_n$ are random variables on a common probability space such that $\text{Var}(X_i) = \text{Var}(Y_i) = 4$ for all i, and

$$\rho_{X_i,Y_j} = \begin{cases} 3/4 & \text{if } i = j \\ -1/4 & \text{if } |i-j| = 1 \\ 0 & \text{else.} \end{cases}$$

   Let $W = \sum_{i=1}^n X_i$ and $Z = \sum_{i=1}^n Y_i$. Express $\text{Cov}(W, Z)$ as a function of n.

6. **[15 points]** Suppose X and Y have a bivariate Gaussian joint distribution with $E[X] = E[Y] = 0$ and $\text{Var}(X) = 1$. (The variance of Y and the correlation coefficient are not given.) Finally, suppose X is independent of $X + Y$.

   (a) **[5 points]** Find $\text{Cov}(X, Y)$.

   (b) **[5 points]** Find $E[X|X + Y = 2]$.

   (c) **[5 points]** Find $E[Y|X = 2]$.

7. **[15 points]** Consider a standard deck of 52 cards, and choose one card at random. Let X take the following values based on the rank of the card chosen:

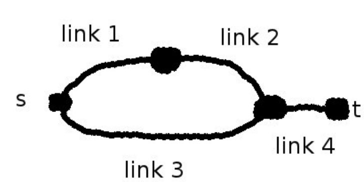
$$X = \begin{cases} 0 & \text{rank is } A, K, Q, J \\ 1 & \text{rank is } 10, 9, 8, 7, 6 \\ 2 & \text{rank is } 5, 4, 3 \\ 3 & \text{rank is } 2. \end{cases}$$

   (a) **[5 points]** Find the pmf of X.

   (b) **[5 points]** Find $E[X]$.

   (c) **[5 points]** Find $E[\sin(X\pi/2)]$.

8. **[20 points]** Consider the following $s - t$ flow network, where link $i \in \{1, 2, 3, 4\}$ fails with probability p_i. Let c_i be the flow capacity of link i, then $c_1 = 10$, $c_2 = 20$, $c_3 = 10$, $c_4 = 30$.



   (a) **[5 points]** What values can the capacity C of this network take?

(b) **[10 points]** Find the distribution (pmf) of its capacity in terms of the p_i's. Do *not* rely on the fact that the sum over the pmf is one. That is, for each possible value u of the capacity, find an expression for $p_X(u)$ that is not simply one minus the sum of the other probabilities.

(c) **[5 points]** Let $p_i = 1/2$ for all i. Obtain the numerical values for the pmf of the capacity (simplify as much as you can).

9. **[25 points]** Joe fishes with a net. Joe's strategy is unusual: he waits for two fish to jump, then throws the net immediately after seeing the second fish jump. Fish jump according to a Poisson process with parameter $\lambda = 0.1$ jumps per minute. Assume that Joe begins waiting at time zero; the first fish jumps at time U_1, and the second fish jumps at time T_2. In all parts of this problem, you may leave numerical powers of e in your answer.

   (a) **[7 points]** What's the probability that $T_2 \geq 30$ minutes?

   (b) **[6 points]** What is $E[T_2]$?

   (c) **[6 points]** Suppose that Joe sees the first fish jump at time $U_1 = u$ minutes. What is $f_{T_2|U_1}(t|u)$?

   (d) **[6 points]** Joe's friend doesn't see the first fish jump, but sees Joe throw the net immediately after the second fish jumps at $T_2 = t$ minutes. What is $f_{U_1|T_2}(u|t)$?

10. **[25 points]** Suppose two random variables X and Y have the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \dfrac{uv+1}{4}, & \text{if } -1 \leq u \leq 1 \text{ and } -1 \leq v \leq 1, \quad (1) \\ 0, & \text{elsewhere.} \quad (2) \end{cases}$$

   (a) **[5 points]** Find the numerical value of $P\{Y \leq -\frac{1}{3}\}$.

   (b) **[5 points]** Find the constant estimator δ^* of Y and the corresponding mean square error (MSE).

   (c) **[10 points]** Find the unconstrained estimator $g^*(X)$ for observation X and the corresponding MSE. (Hint: The MSE must be deterministic, not random.)

   (d) **[5 points]** Find the linear estimator $L^*(X)$ for observation X and the corresponding MSE.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

   (a) Suppose X and Y have the bivariate joint Gaussian distribution with $\mu_X = \mu_Y = 0$, $\sigma_X^2 = \sigma_Y^2 = 1$, and $\rho_{X,Y} = 0.5$.

TRUE	FALSE	
☐	☐	$X - Y$ is independent of $X + Y$.
☐	☐	$P\{X + Y \geq 1\} = Q(\sqrt{5})$

   (b) Let X and Y be random variables on the same probability space with finite variance. Evaluate each of the following claims. $*$ denotes convolution.

TRUE	FALSE	
☐	☐	$E[X+Y] = E[X] + E[Y]$ if and only if X and Y are uncorrelated.
☐	☐	$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ if and only if X and Y are uncorrelated.
☐	☐	$f_{X+Y}(u) = f_X(u) * f_Y(u)$ if and only if X and Y are uncorrelated.

(c) Consider a Poisson process with parameter λ events/minute for some $\lambda > 0$. Let X be the number of events in the first minute, let Y be the number of events in the second minute, and let $Z = X + Y$.

TRUE	FALSE	
☐	☐	$p_Z(u) = (p_X * p_Y)(u)$, where $*$ denotes convolution.
☐	☐	$p_X(313) > 0$.
☐	☐	$p_Z(0) < p_X(0)$ for any $\lambda > 0$.

(d) Let $X_1, X_2, \cdots$ be independent Bernoulli random variables with p. For a positive integer k, let $L(k)$ denote the smallest integer satisfying $X_{L(k)} = 1$ and $L(k) > k$.

TRUE	FALSE	
☐	☐	For any $k \geq 1$, $L(k) - k$ is a geometric random variable with parameter p.
☐	☐	For any $k \geq 1$, $L(L(L(k))) - k$ is a negative binomial random variable with parameters 3 and p.