University of Illinois Spring 2012

ECE 313: Hour Exam II

Tuesday April 10, 2012
7:00 p.m. — 8:15 p.m.
100 Material Sciences and Engineering Building & 151 Everitt Laboratory

1. (a) The event in question is equal to the union of 12 events, one for each edge, where the event for an edge is that the nodes at both ends of the edge burn out. So the probability of the event is less than or equal to $12(0.001)^2 = 1.2 \times 10^{-5}$.

   (b) There are $\binom{8}{2}$, equal to 28, ways to choose a pair of corners. The event in question is equal to the union of 28 events, one for each pair of corners, where the event for a pair is that both corners in the pair burn out. So the probability two or more corners burn out is less than or equal to $28(0.001)^2 = 2.8 \times 10^{-5}$.

2. (a) The (duration of time) times (the rate per unit time) is $(24)\lambda = 2.4$.

   (b) The number of crashes in a five hour period has the Poisson distribution with mean $5\lambda = 0.5$, so the probability of exacty three crashes in a five hour period is $\frac{(0.5)^3 e^{-0.5}}{3!} = \frac{e^{-0.5}}{48}$.

   (c) The mean time until a disk crash is $\frac{1}{\lambda}$, or ten hours. So the mean time until three disks have crashed is thirty hours.

3. (a) $P\{Z \geq 4 | B = 1\} = P\{X \geq 4\} = Q\left(\frac{4-2}{\sqrt{4}}\right) = Q(1)$.

   (b) $P\{Z = 4 | B = 1\} = 0$ because Z is a continuous-type random variable.

   (c) $P\{Z \leq 4 | B = 0\} = P\{Y \leq 4\} = \Phi\left(\frac{4-7}{\sqrt{9}}\right) = \Phi(-1) = Q(1)$.

   (d) $P\{Z \geq 4\} = P\{Z \geq 4 | B = 1\}P\{B = 1\} + P\{Z \geq 4 | B = 0\}P\{B = 0\} = Q(1)\frac{2}{3} + (1 - Q(1))\frac{1}{3} = \frac{1}{3}(1 + Q(1))$.

4. Let f_a denote the pdf of X. $f_a(u) = \frac{1}{\frac{2}{a} - \frac{1}{a}} = a$ if $\frac{1}{a} \leq u \leq \frac{2}{a}$, and $f_a(u) = 0$ for all other u. The support of f_a must include the observed value $u = 3$, hence $\frac{1}{a} \leq 3 \leq \frac{2}{a}$, or equivalently $\frac{1}{3} \leq a \leq \frac{2}{3}$. Notice that $f_a = a$ is an increasing function of a so the ML estimate is the largest possible value of a. Therefore, $\hat{a}_{ML} = \frac{2}{3}$.

5. (a) This question is most easily answered using LOTUS, though you can check your answer using the results of part (c) below.

$$E[Y] = E[e^X] = \int_0^\infty \lambda e^u e^{-\lambda u} du = \frac{\lambda}{\lambda - 1}$$

   (b) The support of $f_X(u)$ is $u \in [0, \infty]$. The function $Y = e^X$ maps this domain into the range $v \in [1, \infty]$.

   (c)

$$F_Y(v) = P\{e^X \leq v\} = \int_0^{\ln v} \lambda e^{-\lambda u} du = 1 - e^{-\lambda \ln v} = 1 - v^{-\lambda}$$

6. (a) They are not independent because, for example, the support of $f_{X,Y}$ is not a product set (see Proposition 4.4.3.)

(b) For $0 \leq u \leq 1$, $f_{X,Y}(u,v)$ is zero if $v > 2u$. Thus,

$$f_X(u) = \int_0^{2u} \alpha(u+v^2)dv = \alpha\left(2u^2 + \frac{(2u)^3}{3}\right).$$

for $0 \leq u \leq 1$, and $f_X(u) = 0$ elsewhere.

(c) By integrating $f_X(u)$ from 0 to 1, we have

$$1 = \int_0^1 2\alpha\left(u^2 + \frac{4u^3}{3}\right)du = 2\alpha\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{4}{3}\alpha.$$

Since the right-hand side is one, α must be $3/4$.

(d) The support of $f_X(u)$ is $u \in (0,1]$. Thus, the conditional pdf $f_{Y|X}(v|u)$ is defined for $u \in (0,1]$.