University of Illinois Spring 2011

ECE 313: Midterm Exam II

Monday April 11, 2011
7:00 p.m. — 8:15 p.m.
Sections C and E: 1EVRT 151 in Everitt Lab
Sections D and F: 1GH 100 in Gregory Hall

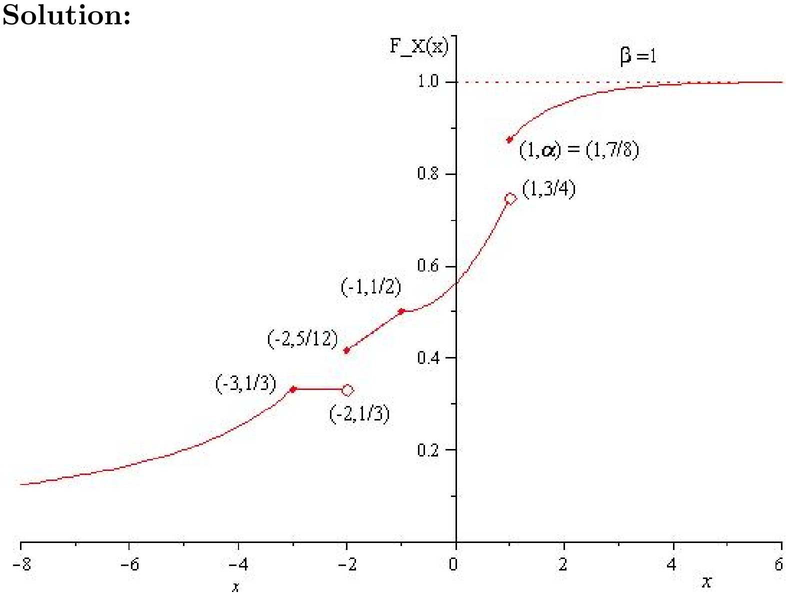
1. **[25 points]** Let X be a random variable with CDF:

$$F_X(x) = \begin{cases} -\frac{1}{x} & x \in (-\infty, -3] \\ \frac{1}{3} & x \in (-3, -2) \\ \frac{1}{12}x + \frac{7}{12} & x \in [-2, -1] \\ \frac{1}{16}(x+1)^2 + \frac{1}{2} & x \in (-1, 1) \\ \alpha & x = 1 \\ \beta - \frac{1}{8}e^{-(x-1)} & x \in (1, \infty) \end{cases}$$

where α, β are real constants.

You must write your answers in the boxes provided, or you will receive zero credit. Show any work in the blank space beside the boxes and if you need extra space use the back of these pages.

(a) **[3 points]** Sketch $F_X(x)$.
Solution:



(b) **[6 points]** Find the values of α and β that make $F_X(x)$ a valid CDF.
Solution: We need $\lim_{x\to\infty} F_X(x) = 1$ so $\boxed{\beta = 1}$. Also, $F_X(x)$ has to be right continuous so $\alpha = F_X(1) = F_X(1^+) = 1 - \frac{1}{8}e^{-(1-1)} = \boxed{7/8 = \alpha}$.

(c) **[2 points]** With the values you just found, is X a continuous random variable? Explain.
Solution: $\boxed{\text{No, it has discontinuities (jumps) at } x = -2 \text{ and } x = 1.}$

(d) **[2 points]** If $x_1 < x_2$, is $F(x_1) < F(x_2)$ for any $x_1, x_2 \in (-\infty, \infty)$?
Solution: $\boxed{\text{No}}$, they can be equal (e.g. $F_X(-2.2) = F_X(-2.1) = 1/3$).

(e) **[2 points]** $\boxed{P(X < -3) \ =}$

Solution: $P(X < -3) = F_X(-3^-) = \boxed{1/3}$.

(f) **[2 points]** $\boxed{P(X = -2) \ =}$

Solution: $P(X = -2) = F_X(-2) - F_X(-2^-) = \frac{5}{12} - \frac{1}{3} = \boxed{1/12}$.

(g) **[3 points]** $\boxed{P(|X| < 1) \ =}$

Solution: $P(|X| < 1) = P(-1 < X < 1) = F_X(1^-) - F_X(-1) = \frac{3}{4} - \frac{1}{2} = \boxed{1/4}$.

(h) **[2 points]** $\boxed{P(X = -2.5) \ =}$

Solution: $P(X = -2.5) = F_X(-2.5) - F_X(-2.5^-) = \frac{1}{3} - \frac{1}{3} = \boxed{0}$.

(i) **[3 points]** $\boxed{E[10] \ =}$

Solution: Using LOTUS: $E[10] = \int_{-\infty}^{\infty}(10)f_X(x)dx = 10\int_{-\infty}^{\infty} f_X(x)dx = \boxed{10}$

2. **[25 points]** This problem concerns binary hypothesis testing with discrete-type observations. ECE graduates from the University of Illinois always receive between one and three job offers.

- Denote by H_0 the hypothesis that a student has taken ECE 313, and by H_1 the hypothesis that that the student has not taken ECE 313. The prior for H_0 is $\pi_0 = 0.3$.
- Students who have taken ECE 313 receive 1,2 and 3 job offers with probability 0.2, 0.3, and 0.5, respectively. Students who have not taken ECE 313 receive 1,2 and 3 job offers with probability 0.7, 0.2, and 0.1, respectively.

The likelihood matrix is therefore given by

	$N = 1$	$N = 2$	$N = 3$
H_0	0.2	0.3	0.5
H_1	0.7	0.2	0.1

(a) **[4 points]** How many possible decision rules are there for this problem? *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Number of decision rules =

Solution: $2^3 = 8$

(b) **[4 points]** Indicate the *maximum likelihood (ML) decision rule* by circling the appropriate entries in the likelihood matrix below.

	$N = 1$	$N = 2$	$N = 3$
H_0	0.2	0.3	0.5
H_1	0.7	0.2	0.1

Solution:

	$N = 1$	$N = 2$	$N = 3$
H_0	0.2	**0.3**	**0.5**
H_1	**0.7**	0.2	0.1

(c) [**4 points**] Indicate the *Maximum a posteriori (MAP) decision rule* by circling the appropriate entries in the likelihood matrix below.

	$N = 1$	$N = 2$	$N = 3$
H_0	0.2	0.3	0.5
H_1	0.7	0.2	0.1

Solution: The table below shows Joint Probability Matrix, with shaded entries that correspond to the MAP rule.

	$N = 1$	$N = 2$	$N = 3$
H_0	0.06	0.09	**0.15**
H_1	**0.49**	**0.14**	0.07

PROBLEM 2, CONTINUED

(d) [**4 points**] What is the false-alarm probability P_{FA} under the ML decision rule? *You must write your numerical answer in the box, or you will receive zero credit for this part. Show any work in the blank space beside the box.*

$P_{FA} =$

Solution: Under the ML decision rule, a false alarm occurs when H_0 is true and $N = 1$ giving $P_{FA} = P\{N = 1 \mid H_0\} = 0.2$

(e) [**4 points**] What is the missed-detection probability P_{MD} under the ML decision rule? *You must write your numerical answer in the box, or you will receive zero credit for this part. Show any work in the blank space beside the box.*

$P_{MD} =$

Solution: Under the ML decision rule, a missed detection occurs when H_1 is true and $N = 2$ or $N = 3$, giving $P_{MD} = P\{N = 2 \mid H_1\} + P\{N = 3 \mid H_1\} = 0.2 + 0.1 = 0.3$

(f) [**5 points**] What is the unconditional probability of error under the ML decision rule? *You must write your numerical answer in the box, or you will receive zero credit for this part. Show any work in the blank space beside the box.*

$P_{error} =$

Solution: The unconditional probability of error is given by $P_{FA}\pi_0 + P_{MD}(1 - \pi_0) = 0.2 \times 0.3 + 0.3 \times 0.7 = 0.06 + 0.21 = 0.27$

3. [**25 points**] This problem concerns Poisson processes.

Transmitters A and B independently send messages to a single receiver in a Poisson manner with average message arrival rates of $\lambda_A = 0.2/s$ and $\lambda_B = 0.6/s$, respectively. All messages are so brief that one may assume that they occupy only single points in time.

(a) [**5 points**] What is the expected value of the time interval between message number three and message number four? *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Expected value = $1/(\lambda_A + \lambda_B) = 5/4\, s$.

Solution: The rate of arrivals is $\lambda_A + \lambda_B$. Since the inter-arrival time has a geometric distribution with a parameter equal to the rate of arrivals, the expected value of interest equals $1/(\lambda_A + \lambda_B)$.

(b) **[5 points]** What is the expected value of the number of type A messages that arrived in a time interval of length $12s$? *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Expected value = $\lambda_A\, t = 2.4$

Solution: The number of arrivals of type A processes is Poisson distributed, with parameter $\lambda_A\, t$.

(c) [**5 points**] What is the probability that, during an interval of duration $t = 10s$, a total of exactly nine messages will be received? *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Probability = $e^{-8}8^9/9!$.

Solution: The expected number of arrivals in $10s$ equals $0.8 \times 10 = 8$. The number of arrivals is Poisson distributed.

(d) [**10 points**] Now, assume that the number of bits in every message, regardless of its transmitting source, is a random variable W which takes the values 1 and 2, both with probability 1/2. Let N be the total number of bits received during an interval of duration t. Determine the expected value of the random variable N. *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Probability = $1.2\,t$.

Solution: We will use the fact that the expected value of a sum of M identically distributed random variables X_i, $i = 1, 2, \ldots$ equals $E[\sum_{i=1}^{M} X_i = E[X_1]E[M]$. The expectations of interest are $E[X_1] = 3/2$ and $E[M] = 0.8t$.

4. **[25 points]** This problem concerns Gaussian Random Variables.

   (a) **[5 points]** Let $f(u)$ be a pdf function, given by

$$f(u) = \frac{K}{\sqrt{2\pi}} \exp(-u^2/2), \quad u \geq 0, \tag{1}$$

   and zero otherwise. Find the value of the constant K.
   You must write your numerical answer in the box, or you will receive zero credit for this part.

K = 2

Solution: The pdf is that of a Gaussian random variable, but restricted to positive values only. Hence, the constant has to be equal to two.

Now, let X be a Gaussian random variable with mean 2 and variance 4.

   (b) **[5 points]** Find $P\{|X - 3| + 1 < 0\}$ *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Probability = 0

Solution: This is an impossible event. Hence, the probability equals to zero.

PROBLEM 4, CONTINUED

(c) [**5 points**] Find $P\{X^2 \leq 1\}$ *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Probability = 0.247

Solution: Observe that

$$P\{X^2 \leq 1\} = P\{-1 \leq X \leq 1\} = P\{(-1-2)/2 \leq (X-2)/2 \leq (1-2)/2\} = \Phi(-1/2) - \Phi(1/2) = 0.247. \tag{2}$$

The pdf is that of a Gaussian random variable, but restricted to positive values only. Hence, the constant has to be equal to two.

(d) [**10 points**] Using the DeMoivre-Laplace approximation, determine the probability that *more* than 80 percent of the students in this class (assuming that there are 200 students enrolled) will solve the first question in this problem. Assumed that everyone arrives at the solution independently, and that the probability of a correct answer is 0.9 and of a wrong answer 0.1. *You must write your numerical answer in the box, or you will receive zero credit for this part.*

Probability = 1

Solution: The easiest way to show this result is to say that the Gaussian approximation variable has mean $np = 180$ and variance $npq = 18$. Hence, the standard deviation is roughly 4.5. You are looking for $P\{X \geq 160\}$ which is greater than the probability in the three-sigma range around the probability. From the homeworks, you know that this range contains roughly 98 percent of the probability mass. Hence, it is safe to say that the probability is close to 1.