University of Illinois Spring 2011

ECE 313: Final Exam

Friday May 13, 2011
1:30 p.m. — 4:30 p.m.
Rm. 100 Noyes Lab

Name: (in BLOCK CAPITALS) ______________________________

University ID Number: ____________________

Signature: ______________________________

Section: ☐ C: MWF 10am ☐ D: MWF 11am ☐ E: TuTh 12:30pm ☐ F: MWF 12pm

This exam is closed book and closed notes except that two 8.5"×11" sheet of notes are permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

Grading

1. 16 points ____________
2. 6 points ____________
3. 8 points ____________
4. 15 points ____________
5. 16 points ____________
6. 16 points ____________
7. 13 points ____________
8. 10 points ____________

Total (100 points) ____________

1. **[16 points]** For this problem, Bob flips a fair coin multiple times and N denotes the number of heads that occur. Mary rolls fair dice, and S denotes the number of sixes she rolls.

   For each part below, **circle the letter of the correct answer and fill in the blanks for your selection.** If you circle more than one answer or if you do not circle an answer, you will receive zero credit. NOTE: The problem parts are independent.

   (a) Suppose Bob flips the coin 7 times. What type of random variable is N?

   A. Poisson, $\lambda =$ _____ B. Geometric, $p =$ _____

   C. Binomial, $n =$ _____ $p =$ _____ D. Bernoulli, $p =$ _____

   E. Negative Binomial, $r =$ _____ $p =$ _____ F. Exponential, $\lambda =$ _____

   (b) Mary rolls N dice. The conditional pmf $p_{S|N}$ has which of the following distributions?

   A. Poisson, $\lambda =$ _____ B. Geometric, $p =$ _____

   C. Binomial, $n =$ _____ $p =$ _____ D. Bernoulli, $p =$ _____

   E. Negative Binomial, $r =$ _____ $p =$ _____ F. Exponential, $\lambda =$ _____

   (c) Suppose Bob flips his coin at the rate of 5 coin flips per minute. Let M denote the number of heads that occur in the first 2 minutes. What type of random variable is M?

   A. Poisson, $\lambda =$ _____ B. Geometric, $p =$ _____

   C. Binomial, $n =$ _____ $p =$ _____ D. Bernoulli, $p =$ _____

   E. Negative Binomial, $r =$ _____ $p =$ _____ F. Exponential, $\lambda =$ _____

   (d) Mary rolls N dice. The random variable Y takes the value one if $S = N$, and zero otherwise. What kind of random variable is Y?

   A. Poisson, $\lambda =$ _____ B. Geometric, $p =$ _____

   C. Binomial, $n =$ _____ $p =$ _____ D. Bernoulli, $p =$ _____

   E. Negative Binomial, $r =$ _____ $p =$ _____ F. Exponential, $\lambda =$ _____

(e) Bob flips his coin until a heads occurs. Let Q denote the number of coin flips. What kind of random variable is Q?

A. Poisson, $\lambda =$ _____

B. Geometric, $p =$ _____

C. Binomial, $n =$ _____ $p =$ _____

D. Bernoulli, $p =$ _____

E. Negative Binomial, $r =$ _____ $p =$ _____

F. Exponential, $\lambda =$ _____

(f) Mary rolls N dice simultaneously until she simultaneously rolls N sixes N times. (For example, if $N = 3$, Mary rolls three dice simultaneously until she simultaneously rolls three sixes on three separate simultaneous rolls.) Let W be the number of times she simultaneously rolls the dice. What kind of random variable is W?

A. Poisson, $\lambda =$ _____

B. Geometric, $p =$ _____

C. Binomial, $n =$ _____ $p =$ _____

D. Bernoulli, $p =$ _____

E. Negative Binomial, $r =$ _____ $p =$ _____

F. Exponential, $\lambda =$ _____

(g) Suppose Bob flips his coin at the rate of 5 coin flips per minute. Let T denote the time at which the first heads occurs. What type of random variable is T?

A. Poisson, $\lambda =$ _____

B. Geometric, $p =$ _____

C. Binomial, $n =$ _____ $p =$ _____

D. Bernoulli, $p =$ _____

E. Negative Binomial, $r =$ _____ $p =$ _____

F. Exponential, $\lambda =$ _____

(h) Suppose Bob flips his coin at the rate of 5 coin flips per minute and that he flips his coin for five minutes without observing any heads. Let K denote the number of heads observed in the sixth minute. What type of random variable is K?

A. Poisson, $\lambda =$ _____

B. Geometric, $p =$ _____

C. Binomial, $n =$ _____ $p =$ _____

D. Bernoulli, $p =$ _____

E. Negative Binomial, $r =$ _____ $p =$ _____

F. Exponential, $\lambda =$ _____

2. **[6 points]** Let X be a random variable whose pdf is given by f_i when hypothesis H_i is true, for $i = 0, 1$, where the f_i are given by

$$f_0(u) = \begin{cases} 2u & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_1(u) = \begin{cases} 1/2 & -1 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) In the box below, write the maximum likelihood decision rule in terms of a single threshold on the observed value of X. Show your work in the space below the box.

ML Decision Rule:

(b) In the box below, write the value of p_{miss} for the ML decision rule.

p_{miss}:

(c) Suppose $\pi_0 = 2/3$. In the box below, write the value of the average error probability, p_e, for the ML decision rule.

p_e:

3. **[8 points]**

Box 1 contains two black balls two white balls. Box 2 contains two green balls and three white balls. A ball is drawn at random from Box 1 and placed in Box 2. A ball is then drawn at random from Box 2. **You must write your answers in the boxes below, or you will receive zero credit.** Show your work in the space provided (not in the boxes).

(a) What is the probability that the ball drawn from Box 2 is black?

P{ ball drawn from Box 2 is black } =

(b) What is the probability that the ball drawn from Box 2 is green?

P{ ball drawn from Box 2 is green } =

(c) Suppose that a green ball is drawn from Box 2. What is the probability that the ball drawn from Box 1 is black?

P{ ball drawn from Box 1 is black | ball drawn from Box 2 is green } =

(d) Suppose that a black ball is drawn from Box 2. What is the probability that the ball drawn from Box 1 is black?

P{ ball drawn from Box 1 is black | ball drawn from Box 2 is black } =

4. **[15 points]** Let X be a geometric random variable with parameter p, and let Y be be a geometric random variable with parameter q. Assume that X and Y are independent.

You must write your answers in the boxes provided, or you will receive zero credit. Show any work in the blank space beside the boxes and if you need extra space use the back of these pages.

(a) Suppose (only for this part of the problem) that $E[X^2] = 15$ and that $Var(X+Y) = 8$. Find p and q.

$p =$
$q =$

(b) Suppose (only for this part of the problem) that $p = 2/3$ and that $q = 1/4$. Find $E[2X+3Y-1]$ and $Var(2X+3Y-1)$.

$E[2X+3Y-1] =$
$Var(2X+3Y-1) =$

(c) Suppose (only for this part of the problem) that $p = 2/3$ and that $q = 1/4$. Find $Cov(2X, 3Y)$.

$Cov(2X, 3Y) =$

(d) Find the joint pmf $p_{X,Y}(i, j)$ for all i and j, and express it in terms of p and q.

$p_{X,Y}(i, j) =$

(e) Find the conditional pmf $p_{X|Y}(i|j)$ for all i and j, and express it in terms of p and q.

$p_{X|Y}(i|j) =$

(f) Suppose that you don't know p but you perform the experiment once and observe that $X^2 = 16$. Find the maximum likelihood estimate $\hat{p}_{ML}$.

$\hat{p}_{ML} =$

5. **[16 points]** Let X and Y be continuous random variables with joint pdf given by $f_{X,Y}(x, y) = 2e^{-2x-y}$ for $x \geq 0, y \geq 0$, and zero otherwise.

You must write your answers in the boxes provided, or you will receive zero credit. Show any work in the blank space beside the boxes and if you need extra space use the back of these pages.

(a) Is this a valid joint pdf? If it is not a valid joint pdf, state one property which is not satisfied. If it is not a valid joint pdf, indicate if there exists a constant c such that $cf_{X,Y}(x, y)$ is a valid density, and if so, find the value of c.

Valid joint pdf (yes/no) ? =

Value of c (if applicable) =

Reason why not valid joint pdf (if applicable) =

(b) Are X and Y independent? If not, state at a reason why not.

Independent (yes/no) ? =

Reason why not independent (if applicable) =

(c) Find the cumulative distribution function (CDF) of X for all x.

$F_X(x) =$

(d) Find the conditional density $f_{Y|X}(y|1)$ for all y.

$f_{Y|X}(y|1) =$

(e) Find the probability that $(X, Y) \in T$, where T is the triangle whose vertices are $(1, 1)$, $(2, 1)$ and $(2, 2)$.

$P\{(X, Y) \in R\} =$

(f) Find the unconstrained optimal minimum mean squared error estimator of Y based on X, denoted by $g^*(X)$.

$g^*(X) =$

6. **[16 points]** This problem is concerned with counting permutations and combinations.

You must write your answers in the boxes provided, or you will receive zero credit. Show any work in the blank space beside the boxes and if you need extra space use the back of these pages.

(a) In how many ways can five people fill five different government positions?

Number of ways =

(b) In how many ways can five people fill the positions if person 1 can never be president and person 2 can never be the vice president? Note that in any government, they can be only one president and only one vice president.

Number of ways with restrictions on person 1 and 2 =

(c) In how many distinct ways can you arrange the letters in the word "banana"?

Number of ways =

(d) In how many ways can you form three pairs if you have a set of five women and a set of six men? Here, a pair consists of one man and one woman.

Number of ways =

(e) In how many ways can you choose a work team with three women and three men, if you have a set of eight women and a set of six men and if 2 given men refuse to work together?

Number of ways =

(f) Assume that you have five electrons and three orbits. In how many ways can you distribute the electrons into orbits so that no orbit contains more than two electrons?

Number of distributions =

7. [**13 points**] This problem is concerned with Poisson processes.

You must write your answers in the boxes provided, or you will receive zero credit. Show any work in the blank space beside the boxes and if you need extra space use the back of these pages.

(a) Cars are passing a point of a road according to a Poisson process. The mean time interval between the cars is 10 min. A hitchhiker arrives to the roadside point at a random instant of time. What is the mean waiting time until the next car arrives?

Mean waiting time =

(b) An employee in a call center works from 8 a.m. until 5 p.m., with breaks between 10.30-10.45, 12.30-13.30 and 14.45-15.00. Assume that calls come in according to a Poisson process with expected number of calls per hour equal to six.

- a) What is the probability that there are at most 10 calls during the breaks?
- b) What is the probability that the first call of the day is after 8.10 a.m.?
- c) What is the probability that the employee can do something else for 45 minutes without being disturbed by a call?

Probability in a) =

Probability in b) =

Probability in c) =

8. **[10 points]** This problem is concerned with minimum mean squared error estimation.

You must write your answers in the boxes provided, or you will receive zero credit. Show any work in the blank space beside the boxes and if you need extra space use the back of these pages.

(a) Assume that you have a fair coin and that you toss it once. If the outcome is tails, you draw a number according to a Gaussian distribution with mean 2 and variance 3. If the outcome is heads, you draw a number according to a Gaussian distribution with mean 1 and variance 2. At the end of the experiment, let Y be the random variable representing the outcome of your drawing (experiment). What is the minimum mean square error *constant estimator* of the random variable Y?

MMSE constant estimator =

(b) Let X be a uniformly distributed random variable in the interval $[1, a]$, where $a > 1$ is finite. Find the maximum likelihood estimator for a given that you observed a realization of X that has value 3.5.

Maximum likelihood estimator =

(c) Write down the equation for the minimum mean square error linear estimator and the minimum mean square error unrestricted estimator.

MMSE linear =
MMSE unrestricted =