University of Illinois Spring 2010

ECE 313: Problem Set 11

Hazard Rate; Decision Making; Joint Distributions of Random Variables

Due: Wednesday April 21 at 4 p.m.

Reading: Ross, Chapters 5 & 6; PowerPoint Lecture Slides 28-33

Noncredit Exercises: Chapter 5: Problems 37-41
Theoretical Exercises 18-21, 29, 30; Self-Test Problems 14, 16. 20.

Noncredit Exercises: Chapter 6: Problems 1-3, 9, 10, 13, 15, 19-23, 40-42
Theoretical Exercises 4, 6; Self-Test Problems 3, 5-7.

This Problem Set contains seven problems.

1. Some systems contain two identical components and have the property that the system works as long as at least one of the components is working. The system is put into operation at $t = 0$. Let X_1 and X_2 denote the time of failure of the two components, and let Y denote the time of failure of the system.

(a) The occurrence of the event $\{Y > T\}$ means that the system is working at time T. Express $\{Y > T\}$ in terms of the events $\{X_1 > T\}$ and $\{X_2 > T\}$. Assume that the latter two events are independent, and express $P\{Y > T\}$ in terms of the probabilities of these events.

(b) Now suppose that $P\{X_1 > T\} = P\{X_2 > T\} = \exp(-\lambda T)$, that is, X_1 and X_2 are exponential random variables with parameter λ.

i. Use the result $E[Y] = \int_0^\infty P\{Y > T\}dT$ to find E[Y], the average lifetime of the system. The average lifetime is also known as the mean time before failure (MTBF) or mean time to failure (MTTF) in the reliability literature.

ii. Find the median value of Y by solving the equation $P\{Y > T\} = ½$ for T. The median value is sometimes called the half-life of the system.

iii. Find the hazard rate of the system.

(c). Compare your answers in parts (b)(i)-(iii) to MTBF λ^{-1}, the median lifetime $\lambda^{-1}\ln 2$, and the hazard rate λ for a system with only one component instead of two.

2. If hypothesis H_0 is true, the pdf of X is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of X is exponential with parameter 10.

(a) Sketch the two pdfs.

(b) State the maximum-likelihood decision rule in terms of a threshold test on the observed value u of the random variable X instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_o(u)$ to 1.

(c) Determine the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part(b).

(d) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to π_o/π_1. Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable X.

(e) If $\pi_o = 1/3$, determine is the average probability of error of the Bayesian decision rule.

(f) What is the average error probability of a decision rule that always decides H_1 is the true hypothesis, regardless of the value taken on by X?
(g) Show that if $\pi_o > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by X. Determine the average probability of error for the maximum-likelihood rule when $\pi_o > 2/3$.

3. The joint pmf $p_{X,Y}(u,v)$ of X and Y is shown.

u:	0	1	3	5
v				
4	0	1/12	1/6	1/12
3	1/6	1/12	0	1/12
-1	1/12	1/6	1/12	0

(a) Determine the marginal pmfs $p_X(u)$ and $p_Y(v)$.
(b) Are X and Y independent random variables?
(c) Determine $P\{X \leq Y\}$ and $P\{X + Y \leq 4\}$.
(d) Determine $p_{X,Y}(u \mid 3)$, $E[X \mid Y = 3]$ and $\text{var}(X \mid Y = 3)$.

4. Let X and Z be independent geometric random variables, having probability mass functions with parameters p and q, respectively. Let $Y = XZ$.
(a) Compute the conditional pmf $p_{Y \mid X}$ and using that compute pmf p_Y .
(b) Compute for each $n \geq 1$, the conditional mean and variance of Y given $X = n$.

5. Let (X,Y) be a pair of continuous random variables with the joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} u+v, 0 \leq u \leq 1, 0 \leq v \leq 1 \\ 0, \qquad \text{otherwise} \end{cases}$$

Determine the value of $P\{2Y < X\}$

6. Let (X,Y) be a pair of continuous random variables with the joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} e^{-u}, 0 < u < v < \infty \\ 0, \qquad \text{otherwise} \end{cases}$$

Let Z=Y/X. Determine the value of $P\{Z \leq a\}$ where a some fixed number in the range $0 < a < 1$.

7. Let (X,Y) be a pair of continuous random variables with the joint pdf given by

$$f_{X,Y}(u,t) = \begin{cases} C_o(u+t), u \geq 0, t \geq 0 \text{ and } u+t \leq 2 \\ 0, \qquad \text{otherwise} \end{cases}$$

(a) Evaluate the constant C_o.
(b) Let $Z = \min\{2X, Y\}$. Determine is the pdf of Z.
(c) Let $W = XY$. Determine the pdf of W.