University of Illinois Spring 2010

ECE 313: Problem Set 10

Due: Wednesday, April 7 at 4 p.m.
Reading: Ross, Chapter 5; Lecture Notes 26-28.
Noncredit Exercises: DO NOT turn these in.
Chapter 5: Problems 11,15 and 31;
Theoretical Exercises 18,30.

This Problem Set contains seven problems.

1. **[Moments of random variables]**
   Theoretical Exercises, Chapter 5 of the textbook, Problem 5.5.

2. **[Gaussian Random Variables]**
   Let X be a Gaussian random variable with mean $\mu = -1$ and variance $\sigma^2 = 4$.

   (a) Find the mean and variance of $2X + 5$.

   Let $\Phi(x)$ denote the CDF of a standard Gaussian random variable, and let $Q(x) = 1 - \Phi(x)$. Suppose that Calculator A can evaluate only $\Phi(x)$ and only for nonnegative values of x. On the other hand, suppose that Calculator B can evaluate only $Q(x)$, again only for $x \geq 0$. Both calculators can perform standard functions, like addition and multiplication. For each of the probabilities in parts (**b**) through (**e**), write down *two* alternative expressions: one for evaluation using Calculator A, and the other for evaluation using Calculator B.

   (b) $P(X < 0)$
   (c) $P(-10 < X < 5)$
   (d) $P(|X| \geq 5)$
   (e) $P(X^2 - 3X + 2 < 0)$

3. **[Gaussian approximation of Binomial random variables]**
   Repeat the derivations of Example 6f on p. 154 of the textbook (seventh edition)/ Example 6f on p. 137 of the textbook (eighth edition) using the binomial approximation to the normal distribution. How accurate is the approximation?

4. **[Generation of random variables with prescribed CDF]**
   Suppose you are writing a simulation program, and you'd like to generate a random variable X with the following probability density function:

   $$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{else.} \end{cases}$$

   A call to the random number generator produces a variable U that is uniformly distributed on the interval $[0, 1]$. (At least this is true in theory, in reality computers use a "pseudo-random" number generator.) By applying a function g to U, a new random variable $X = g(U)$ can be produced. Find the increasing function g on the interval $[0, 1]$ such that $X = g(U)$ has the specified probability density. (Hint: Begin by finding the cumulative distribution function F_X.)

5. **[Function of one random variable]**
A capacitor is discharged for a random interval of time Y from an initial voltage X which is also random variable independent of Y. The time constant of the circuit is $1/3$ and thus the voltage across capacitor at the end of the time interval Y is $Z = Xe^{-3Y}$. If X has mean 1 and the variance 4 and Y is an exponential random variable with parameter 3, find the mean and the variance of Z.

6. **[Function of one random variable]**
Consider a sphere whose radius is a random variable $\mathcal{R}$ with pdf $f_{\mathcal{R}}(u) = 2u,\ 0 < u < 1$, and 0 otherwise.

   (a) What is the average radius of the sphere? What is the average volume? What is the average surface area? If a sphere of average radius is called an *average sphere*, then does an average sphere have the average volume? Does it have the average surface area?

   (b) Show that $\mathsf{E}[\mathcal{R}] > \mathsf{E}[\mathcal{R}^2] > \mathsf{E}[\mathcal{R}^3]$ for *any* pdf for $\mathcal{R}$ that is nonzero only on the unit interval $(0, 1)$.

7. **[Function of one random variable]**
Let X be a standard Gaussian random variable. Find the CDF and pdf of the random variable $Y = X^2$.