University of Illinois **Spring 2010**

ECE 313: Problem Set 9
Probability Density, Poisson Processes

Due:	Wednesday March 31 at 4 p.m.
Reading:	Ross, Chapter 5; Powerpoint Lecture Slides, Sets 22-25
Noncredit Exercises:	**Chapter 5:** Problems 1-3, 5, 6, 15-19, 23-25, 32-34; Theoretical Exercises 1, 8; Self-Test Problems 1-4

1. [**Probability Density**]
Random variable X has the following PDF:

$$f_X(u) = \begin{cases} \frac{1}{5} & 0 < u \leq 1 \\ \frac{1}{2} & 1 < u \leq 2 \\ \frac{1}{10} & 2 < u \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Answer each of the following questions by shading and labeling the area of a region on a graph. **Do not** perform any integrals.

   (a) What is $P(2 < X \leq 3)$?

   (b) What is $P(-1 \leq X \leq 1.5)$?

   (c) What is $P(2 \leq X \leq 6)$?

   (d) What is $P(1.173 \leq X \leq 1.174)$?

   (e) What is $F_X(v)$?

2. [**Black Swans**]
A recent census conducted in the country of Metasylvania found that the body weight of adult Metasylvanians, measured in pounds, is a random variable X with the following probability density function:

$$f_X(u) = \begin{cases} Ae^{-0.02(u-80)} & 80 \leq u \\ 0 & \text{else} \end{cases}$$

The same survey found that the personal wealth of adult Metasylvanians, measured in Metasylvanian dollars, is a random variable Y with the following probability density function:

$$f_Y(u) = \begin{cases} B\left(\frac{u}{100}\right)^{-1.5} & 100 \leq u \\ 0 & \text{else} \end{cases}$$

   (a) Find the constants A and B.

   (b) Notice that all Metasylvanians weigh at least 80 pounds.

      i. What is the probability that an individual Metasylvanian weighs at least 160 pounds?

      ii. What is the probability that an individual Metasylvanian weighs at least 800 pounds?

      iii. What is the expected weight of a Metasylvanian?

   (c) Notice that every Metasylvanian has a personal wealth of at least \$100.

      i. What is the probability that a Metasylvanian has a personal wealth of greater than \$200?

      ii. What is the probability that a Metasylvanian has a personal wealth of greater than \$1000?

      iii. What is the probability that a Metasylvanian has a personal wealth of greater than \$100,000,000?

      iv. Prove that the expected personal wealth of a Metasylvanian is infinite.

      v. **Optional**: why is extreme wealth so much more common than extreme obesity?

3. **[Expectation]**

   (a)
   $$f_Y(u) = \begin{cases} 1 - |u| & -1 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

   Find $E[Y]$, $E[|Y|]$, $E[Y^2]$, and σ_Y^2.

   (b) Random variable Z is the absolute value of random variable Y ($Z = |Y|$).

      i. Find $F_Z(v)$, the CDF of random variable Z.

      ii. Find $f_Z(u)$, the PDF of random variable Z.

      iii. Find $E[Z]$, $E[|Z|]$, $E[Z^2]$, and σ_Z^2 without performing any new integrals (hint: look at your answers to part (a)).

4. **[Using LOTUS]**
   Let X be a continuous random variable that is uniformly distributed on $[-1, +1]$.

   (a) Let $Y = X^2$. Calculate the mean and the variance of Y.

   (b) Let $Z = g(X)$ where $g(u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0. \end{cases}$ Find $\mathsf{E}[Z]$.

5. **[Two-Sided Exponential PDF]**
   Suppose X has the following PDF:
   $$f_X(u) = \begin{cases} 0.1e^{u/4} & -\infty < u \leq 0 \\ 0.1e^{-u/6} & 0 \leq u < \infty \end{cases}$$

   Find $E[X]$, $E[X^2]$, and σ_X^2.

6. **[Gamma PDF]**
   Buses arrive at the corner of Sixth and Green at an average rate of λ buses per second. Arrivals form a Poisson process. Let X be the number of buses that arrive in t seconds; X is distributed as
   $$p_X(u) = \begin{cases} e^{-\lambda t}\frac{(\lambda t)^u}{u!} & 0 \leq u \\ 0 & \text{else} \end{cases}$$

   Let Y be the amount of time that you must wait for the 3^{rd} bus to arrive. Notice that the event $\{X < 3\}$ (fewer than three buses arrive in t seconds) is exactly equivalent to the event $\{Y > t\}$ (the waiting time for the third bus is greater than t seconds).

   (a) Express $1 - F_Y(t)$, the complementary CDF of variable Y, in terms of $p_X(u)$.

   (b) Differentiate your answer to part (a) with respect to t in order to find the PDF of variable Y. Demonstrate that your result is equal to a Gamma PDF.

   (c) The k^{th} arrival time in a Poisson process is the sum of k independent first-order arrival times. Take advantage of this fact to find $E[Y]$ and σ_Y^2.

7. **[Poisson Bus Lines]**
   Buses arrivals at Sixth and Green form a Poisson process, with an average arrival rate of λ buses per minute. Buses arriving after 5:00PM always arrive in the following order: the Turquoise line (#16) always comes first, followed by the Fuchsia line (#102), followed by the Octarine (#42), followed by buses of other colors. Your bus is the Octarine. Unfortunately, there is only one Octarine bus per day; if you miss it, you will have to walk home!

   (a) You arrive at the bus stop at exactly 5:00.

      i. What is the probability that you will wait between t and $t+dt$ minutes for your bus, assuming that dt is very small?

ii. What is your expected waiting time?

iii. What is the variance of your waiting time?

(b) Your 4:00 lecturer (being half elvish) has a poor grasp of time, therefore you reach the bus stop at 5:05PM. Fortunately, your friend Joe has been at the bus stop since 5:00. He says you haven't missed your bus; the Turquoise bus came at 5:03, but the Fuchsia and Octarine buses have not arrived yet. Assuming that Joe is reliable, what is the probability that you will wait between τ and $\tau + dt$ minutes for your bus?

(c) Aja has also been at the bus stop since 5:00, but she disagrees with Joe. She believes that both the Turquoise and Fuchsia buses have come and gone, but that the Octarine bus has not yet arrived. Based on your knowledge of their honesty, reliability, and observational skills, you suppose that Aja is correct with probability 0.75, and Joe is correct with probability 0.25.

i. What is the probability that you will wait between τ and $\tau + dt$ minutes for your bus to arrive?

ii. What is your expected waiting time?

iii. What is the variance of your waiting time?