University of Illinois **Spring 2010**

ECE 313: Problem Set 8
CDFs; Continuous Random Variables

Due: Wednesday March 17 at 4 p.m.
Reading: Ross, chapter 5; Powerpoint Lecture Slides, Sets 20-22

1. **[Maximization of the newsboy's profit]**
   Each day, a newsboy buys newspapers from the publisher for c_1 cents each, sells them for c_2 cents each, and recycles the unsold papers (if any) getting c_3 cents for each. Note that $c_2 > c_1 > c_3$. Let H denote the number of papers that the newsboy purchases each day. The demand for papers is a discrete random variable $\mathbb{X}$ that takes on nonnegative integer values. Do *NOT* assume that $\mathbb{X}$ is a binomial random variable. Let $F(u)$ denote the CDF of $\mathbb{X}$.

   (a) Express the probability that the newsboy is able to sell all H papers in terms of $F(u)$.

   (b) One day, the newsboy decides to buy *one* additional paper in the hopes of selling it and increasing his profit. Express the probability that he is unable to sell the additional paper in terms of $F(u)$. Be sure you understand the difference between "not being able to sell the $(H+1)$-th paper" and "being able to sell all H papers but not the extra $(H+1)$-th paper."

   (c) Find $A(H+1)$, the average *additional* profit from the sale of the extra (that is, $(H+1)$-th) paper.

   (d) Use the properties of CDFs to show that $A(H+1)$ is a non-increasing function of H and that $A(H+1) < 0$ for sufficiently large values of H.

   (e) What choice of H *maximizes* the newsboy's average daily profit? Call this value of H as H_0.

2. **[Using CDFs]**
   The number of hours that a student spends on ECE 440 homework is a random variable $\mathbb{X}$ with CDF

$$F_{\mathbb{X}}(u) = \begin{cases} 0, & u < 0, \\ (1+u)/8, & 0 \le u < 1, \\ 1/2, & 1 \le u < 2, \\ (4+u)/8, & 2 \le u < 4, \\ 1, & u \ge 4. \end{cases}$$

   Note that this is a *mixed* random variable: it takes on some values with nonzero probability (like a discrete random variable) but also takes on all values in intervals of the real line (like a continuous random variable). Note also that we are given the CDF.

   (a) Find $P\{\mathbb{X} = 2\}$, $P\{\mathbb{X} < 2\}$, $P\{\mathbb{X} > 2\}$, $P\{1 \le \mathbb{X} \le 3\}$, and $P\{\mathbb{X} > 2 \mid \mathbb{X} > 0\}$.

   (b) Find $\mathsf{E}[\mathbb{X}]$.

3. **[Validity of PDFs]**
   Nine functions $f(u)$ are shown below. Note that in each case, $f(u) = 0$ for all u not in the interval specified. In each case,

   - determine whether $f(u)$ is a valid probability density function (pdf).

- If $f(u)$ is not a valid pdf, determine if there exists a constant C such that $C \cdot f(u)$ is a valid pdf.

(a) $f(u) = 2u, \quad 0 < u < 1.$ (b) $f(u) = |u|, \quad |u| < \frac{1}{2}$
(c) $f(u) = 1 - |u|, \quad |u| < 1,$ (d) $f(u) = \ln u, \quad 0 < u < 1.$ Hint: $\ln u$ can be integrated by parts.
(e) $f(u) = \ln u, \quad 0 < u < 2,$ (f) $f(u) = \frac{2}{3}(u - 1), \quad 0 < u < 3,$
(g) $f(u) = \exp(-2u), \quad u > 0.$ (h) $f(u) = 4\exp(-2u) - \exp(-u), \quad u > 0,$
(i) $f(u) = \exp(-|u|), \quad |u| < 1,$

4. **[Calculating probabilities from pdfs]**
The continuous random variable $\mathbb{X}$ has pdf $f_{\mathbb{X}}(u) = \begin{cases} c(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$
(a) What is the value of c?
(b) Find $P\{\mathbb{X} > -0.5\}$.
(c) Find $P\{6\mathbb{X}^2 > 5\mathbb{X} - 1\}$.

5. **[Valid CDFs]**
Which of the following functions $F(u)$ are valid CDFs? For those that are valid CDFs, compute the probability that the absolute value of the random variable exceeds 0.5.

(a) $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \le u < 1, \\ 1, & u \ge 1. \end{cases}$ (b) $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \le u \le 2, \\ 1, & u > 2. \end{cases}$

(c) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u \le 0, \\ 1 - \frac{1}{4}\exp(-3u), & u > 0, \end{cases}$ (d) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u < 0, \\ 1 - \frac{1}{4}\exp(-3u), & u \ge 0, \end{cases}$