University of Illinois Spring 2010

ECE 313: Problem Set 6

Due: Wednesday, March 3rd at 4 p.m.
Reading: Ross, Chapter 3; Lecture Notes 14-18.
Noncredit Exercises: DO NOT turn these in.
Chapter 3: Problems 80,84 and 86;
Theoretical Exercises 8,16 and 21.

This Problem Set contains seven problems.

1. [**Conditional Probability**]
   Die A has four red and two white faces, whereas die B has two red and four white faces. A fair coin is flipped once. If it falls heads, the game continues by throwing die A alone. If it falls tails, die B is to be used.

   (a) Show that the probability of red at any throw is 1/2.

   (b) If the first throw resulted in red, what is the probability of red in the third throw?

   (c) If red turns up in the first n throws, what is the probability that die A is being used?

2. [**Conditional Probability and Poisson Random Variables**]
   Each customer who enters Laura's clothing store will purchase a suit with probability p. If the number of customers entering the store is Poisson distributed with mean λ, what is the probability that Laura does not sell any suits? What is the probability of her selling k suits?

3. [**Law of total probability**]
   Alice and Bob play the following game. First, Alice rolls a fair die and then Bob rolls the fair die. If Bob rolls a number at least as large as Alice's number, he wins the game. But if Bob rolled a number smaller than Alice's number, then Alice rolls the die again. If her second roll gives her a number that is less than or equal to Bob's number, the game ends with no winner (a tie, or draw as the British call it). If her second roll gives a number larger than Bob's number, Alice wins the game.

   Find the probability that Alice wins the game and the probability that Bob wins the game. Also, find the probability of a tie directly (and not as $P(\text{tie}) = 1 - P(\text{Alice wins}) - P(\text{Bob wins})$.) If the three probabilities do not add up to 1, explain.

4. [**Conditional Expectations**]
   A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that takes him to the mine after three hours of travel. The third door leads into a tunnel that returns him to his mine after five hours. Assuming that the miner is at all times equally likely to choose any of the doors, what is the expected length of time until the miner reaches safety?

5. [**Conditional Variance**]
   Let X be a random variable with mean μ and variance σ^2. Let F be an even of positive probability. Express the conditional variance of $Y = (3X - \mu)/(2\sigma)$ given F in terms of the conditional variance of X given F.

6. **[Decision Making]**
If H_0 is the true hypothesis, the random variable X takes on values 0, 1, 2, and 3 with probabilities 0.1, 0.2, 0.3, and 0.4 respectively. If H_1 is the true hypothesis, the random variable X takes on values 0, 1, 2, and 3 with probabilities 0.4, 0.3, 0.2, and 0.1 respectively.

   (a) Write down the likelihood matrix L and indicate the *maximum-likelihood decision rule* by shading the appropriate entries in L. What is the false-alarm probability P_{FA} and what is the missed-detection probability P_{MD} for the maximum-likelihood decision rule?

   (b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 0.7$ and $\pi_1 = 0.3$. Use the law of total probability to find the average error probability of the maximum-likelihood decision rule that you found in part (a).

   (c) Use the *a priori* probabilities given in part (b) to find the joint probability matrix J and indicate on it the Bayesian decision rule, which is also known as the minimum-error-probability (MEP) or maximum *a posteriori* probability (MAP) decision rule. What is the average error probability of the Bayesian decision rule? Is it smaller or larger than the average error probability of the maximum-likelihood decision rule? In the latter case, provide a brief explanation as to why the minimum-error-probability rule has a larger average error probability than another rule.

7. **[Detection problem for geometric random variables]**
A transmitter chooses one of two routes (Route 0 or Route 1) and repeatedly transmits a packet over the chosen route until the packet is received without error (that is, without CRC checksum failure) at the receiver. X denotes the number of times the packet is transmitted over the chosen route including the final error-free transmission. Assuming that the successive transmissions are independent trials of an experiment, the two hypotheses are

   - H_1: Route 1 is used for packet transmission: $X \sim \text{Geometric}(p_1)$
   - H_0: Route 0 is used for packet transmission: $X \sim \text{Geometric}(p_0)$

   where $0 < p_1 < p_0 < 1$ are the probabilities of error-free transmission over the two routes.

   (a) State the maximum-likelihood decision rule as to which route was used as a threshold test on the observed value of X.

   (b) Suppose the transmitter chooses Route 0 and Route 1 with probabilities π_0 and $\pi_1 = 1 - \pi_0$ respectively, i.e., π_0 and π_1 are the *a priori* probabilities of hypotheses H_0 and H_1. Assume that $0 < \pi_0 < 1$.
   For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_1 regardless of the value of the observation X?
   For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_0 regardless of the value of the observation X?