University of Illinois **Spring 2010**

ECE 313: Problem Set 12
Functions of random variables, conditional pdfs, covariances

Due:	Wednesday, May 5 at 4 p.m.
Reading:	Ross, Chapter 6, Sections 1-4; Powerpoint Lecture Slides, Sets 30-34
Noncredit Exercises:	**Chapter 6:** Problems 1-3, 9, 10, 13, 15, 19-23, 40-42; Theoretical Exercises 4, 6; Self-Test Problems 3, 5, 6, 7

1. [**Joint Distributions**]
Suppose two jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint distribution given by

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} \frac{1}{C}, & \text{if } -\frac{1}{2} \le u \le \frac{1}{2}, -\frac{1}{2} \le v \le \frac{1}{2}, \text{ and } u^2+v^2 > \frac{1}{16} \\ 0, & \text{otherwise} \end{cases}$$

   (a) find C.

   (b) find $f_{\mathbb{X}|\mathbb{Y}}(u|0.45)$, $f_{\mathbb{X}|\mathbb{Y}}(u|0)$, $f_{\mathbb{Y}|\mathbb{X}}(v|0.45)$, and $f_{\mathbb{Y}|\mathbb{X}}(v|0)$. Are $\mathbb{X}$ and $\mathbb{Y}$ independent?

   (c) Let $\mathbb{Z} = \mathbb{X}^2 + \mathbb{Y}^2$. Find $F_{\mathbb{Z}}(\frac{\pi}{64})$.

2. [**Joint Distributions**]
Let $\mathbb{X}$ and $\mathbb{Y}$ be two independent random variables where $\mathbb{X}$ is exponentially distributed of rate λ_1 and $\mathbb{Y}$ is exponentially distributed of rate λ_2.

   (a) Let $\mathbb{Z} = \min(\mathbb{X}, \mathbb{Y})$. Find the density of $\mathbb{Z}$, given by $f_{\mathbb{Z}}(a)$. (*hint: first find $P(\mathbb{Z} > a)$ using equivalence of events*).

   (b) Define $G(c)$ as

$$G(c) = \int_{u=c}^{\infty} \int_{v=c}^{u} \lambda_1 e^{-\lambda_1 u} \lambda_2 e^{-\lambda_2 v} dv du.$$

   Solve for $G(c)$ in closed form.

   (c) Let B be the event $\{\mathbb{X} > \mathbb{Y}\}$. Express $P(B)$ in terms of $G(c)$ for some value of c and find solve for it in closed form.

   (d) Express $P(\mathbb{Z} > c|B)$ in terms of a ratio $G(c_{\text{num}})/G(c_{\text{denom}})$ for some specific values of c_{num} and c_{denom}, solve for it in closed form, and find $f_{\mathbb{Z}|B}(c)$.

   (e) show without any integrals why $f_{\mathbb{Z}|B^c}(c)$ is also exponentially distributed of rate λ.

3. [**Using a joint density**]
Suppose X and Y are jointly continuous random variables distributed over the unit square with the joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} \frac{3u^2}{2} + 2uv & u,v \in [0,1] \\ 0 & \text{else} \end{cases}$$

   (a) Are X and Y independent? Briefly justify your answer.

(b) Calculate $E[X]$.

(c) Calculate the correlation, $E[XY]$.

(d) Calculate the pdf, $f_Y(v)$, of Y. Be sure to specify it for $-\infty < v < \infty$.

(e) Calculate the conditional density $f_{X|Y}(u|v)$. Be sure to indicate what values of v it is well-defined for, and for such v, specify it for $-\infty < u < \infty$.

4. [**Working with independent Gaussian random variables**]
Let X and Y be independent, $N(0,1)$ random variables.

(a) Find $\text{Cov}(3X + 2Y, X + 5Y + 10)$.

(b) Express $P\{X + 4Y \geq 2\}$ in terms of the Q function defined by $Q(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$.

(c) Express $P\{(X - Y)^2 > 9\}$ in terms of the Q function. (Hint: For what values of $X - Y$ is the event true?)

5. [**Circularly symmetric pdfs**]
The joint pdf of $\mathbb{X}$ and $\mathbb{Y}$ is said to be *circularly symmetric about the origin* if $f_{\mathbb{X},\mathbb{Y}}(u,v) = g(r)$ where $r = \sqrt{u^2 + v^2}$ is the distance of the point (u,v) from the origin.

(a) Let $\mathbb{R} = \sqrt{\mathbb{X}^2 + \mathbb{Y}^2}$ denote the distance of the random point $(\mathbb{X},\mathbb{Y})$ from the origin. Note that $\mathbb{R} \geq 0$. For $\rho \geq 0$, express $F_{\mathbb{R}}(\rho) = P\{\mathbb{R} \leq \rho\}$ as a double integral with respect to u and v and then transform this integral to polar coordinates to show that

$$F_{\mathbb{R}}(\rho) = 2\pi \int_{r=0}^{\rho} r \cdot g(r)\, dr \quad \text{and} \quad f_{\mathbb{R}}(\rho) = \begin{cases} 2\pi\rho g(\rho), & \rho \geq 0, \\ 0, & \rho < 0. \end{cases}$$

(b) Now suppose that $\mathbb{X}$ and $\mathbb{Y}$ are *independent* $\mathcal{N}(0,\sigma^2)$ random variables. Verify that their joint pdf has circular symmetry about the origin, and hence deduce that $\mathbb{R}$ has the *Rayleigh* pdf $f_{\mathbb{R}}(\rho) = \frac{\rho}{\sigma^2}\exp\left(-\frac{\rho^2}{2\sigma^2}\right)$, $\rho \geq 0$ (cf. Ross, p. 214 or 277). Also, show that $P\{\mathbb{R} > \rho\} = \exp\left(-\frac{\rho^2}{2\sigma^2}\right)$.
In communications applications, the noise at the output of a bandpass filter with center frequency f_0 Hz can be expressed as $\mathbb{X}(t)\cos(2\pi f_0 t) - \mathbb{Y}(t)\sin(2\pi f_0 t)$ where for each time instant t, $\mathbb{X}(t)$ and $\mathbb{Y}(t)$ are independent $\mathcal{N}(0,\sigma^2)$ random variables. The *noise amplitude* $\mathbb{R}(t) = \sqrt{(\mathbb{X}(t))^2 + (-\mathbb{Y}(t))^2}$ is thus a Rayleigh random variable for each t. Note that $\mathsf{E}[\mathbb{R}^2(t)] = \mathsf{E}[\mathbb{X}^2(t) + \mathbb{Y}^2(t)] = 2\sigma^2$ is the *noise power*.

(c) Let $\sigma^2 = 1$ in part (b). For $\alpha > 0$, sketch on the u-v plane the region $\{|\mathbb{X}| > \alpha, |\mathbb{Y}| > \alpha\}$ and show that it is a subset of the region $\{\mathbb{R} > \sqrt{2}\alpha\}$. Hence conclude that

$$P\{|\mathbb{X}| > \alpha, |\mathbb{Y}| > \alpha\} = 4Q^2(\alpha) < P\{\mathbb{R} > \sqrt{2}\alpha\} = \exp(-\alpha^2)$$

and therefore for $\alpha \geq 0$, $Q(\alpha) \leq \frac{1}{2}\exp(-\alpha^2/2)$, which you also proved in Problem 10.2.

6. [**Drill problem on jointly continuous random variables II**]
The jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint pdf

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} 2\exp(-u-v), & 0 < u < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the u-v plane and indicate on it the region over which $f_{\mathbb{X},\mathbb{Y}}(u,v)$ is nonzero.

(b) Find the marginal pdfs of $\mathbb{X}$ and $\mathbb{Y}$.

(c) Are the random variables $\mathbb{X}$ and $\mathbb{Y}$ independent ?

(d) Find $P\{\mathbb{Y} > 3\mathbb{X}\}$.

(e) For $\alpha > 0$, find $P\{\mathbb{X} + \mathbb{Y} \leq \alpha\}$.

(f) Use the result in part (e) to determine the pdf of the random variable $\mathbb{Z} = \mathbb{X} + \mathbb{Y}$.

7. **[Some moments for a random rectangle]**
Let $A = XY$ denote the area and $L = 2(X + Y)$ the length of the perimeter, of a rectangle with length X and height Y, such that X and Y are independent, and uniformly distributed on the interval $[0, 1]$.

   (a) Find $E[A]$ and $E[L]$.
   (b) Find $\mathrm{Var}(A)$. (Hint: Find $E[A^2]$ first.)
   (c) Find $\mathrm{Var}(L)$.
   (d) Find $\mathrm{Cov}(A, L)$. (Hint: Find $E[AL]$ first.)
   (e) Find the correlation coefficient, $\rho_{A,L}$. (Hint: Should be less than, but fairly close to, one. Why?)