ECE 313: Problem Set 1
Sets, Events, Axioms of Probability and Their Consequences

Due:	Wednesday September 2 at 4 p.m..
Reading:	*Ross* Chapter 1, Sections 1-4; Chapter 2, Sections 1-5
	Powerpoint Lecture Slides, Sets 1-6
Noncredit Exercises:	**Chapter 1:** Problems 1-5, 7, 9;
	Theoretical Exercises 4, 8, 13; Self-Test Problems 1-15.
	Chapter 2: Problems 3, 4, 9, 10, 11-14;
	Theoretical Exercises 1-3, 6, 7, 10, 11, 12, 16, 19, 20; Self-Test Problems 1-8

Yes, the reading and noncredit exercises are the same as in Problem Set 0.

1. **[Subsets of a finite set]**
Let Ω denote a finite set containing the n elements $\omega_1, \omega_2, \ldots, \omega_n$. The *cardinality* (more informally, the *size*) of a subset $A \subset \Omega$ is the number of elements in A, and is denoted as $|A|$.

   (a) Let $n = 4$. List *all* the subsets of Ω in increasing order of size. How many subsets are there? How many of these subsets are *non-empty* subsets?

   (b) If you listed only 14 or 15 subsets in part (a), please re-do part (a), and this time, include the *empty set* $\emptyset$ and/or Ω as subsets of Ω.

   (c) In your answer to part (a) or (b), verify that for each k, $0 \leq k \leq 4$, the *total number* of subsets of size k is the same as the *total number* of subsets of size $4 - k$. Now explain why for n in general, the total number of subsets of size k is the same as the total number of subsets of size $n - k$.

   (d) Each subset A corresponds to a n-bit vector $(x_1, x_2, \ldots, x_n)$ where $x_i = 1$ if $\omega_i \in A$ and $x_i = 0$ if $\omega_i \notin A$. Writing $A \leftrightarrow (x_1, x_2, \ldots, x_n)$ emphasizes the *one-to-one correspondence*: each subset defines a unique n-bit vector, and each n-bit vector defines a unique subset, e.g. with $n = 4$, we have that $\{\omega_1, \omega_3\} \leftrightarrow (1, 0, 1, 0)$.

      i. What n-bit vectors correspond to Ω and to $\emptyset$? What n-bit vector corresponds to A^c?

      ii. If $A \leftrightarrow (x_1, x_2, \ldots, x_n)$, $B \leftrightarrow (y_1, y_2, \ldots, y_n)$, $(A \cup B) \leftrightarrow (z_1, z_2, \ldots, z_n)$ and $(A \cap B) \leftrightarrow (w_1, w_2, \ldots, w_n)$, express the z_i's and w_i's in terms of the x_i's and y_i's. Hint: you may need the logical operators $\vee$ and $\wedge$ that you may have encountered in ECE 290.

      iii. How many different n-bit vectors are there? How many different subsets are there of Ω?

      iv. "They correspond to the nonempty subsets of Ω" Respond as if you are on Jeopardy™: What is the question to which this statement is the answer? How many subsets of Ω are non-empty?

   (e) One definition of $\binom{n}{k}$ is the total number of subsets of size k of a set Ω of size n.

      i. Compute the numerical values of $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}$, and $\binom{4}{0}$. Do the numbers match up with your answers to part (a) or (b)?

      ii. The *total number* of subsets whose *size* is an *even* number can be expressed as $\binom{n}{0} + \binom{n}{2} + \cdots$ What is the last term in this series? Be careful to distinguish between the cases: n is odd, and n is even.

      iii. Write a similar expression for the *total number* of subsets whose *size* is an *odd* number while continuing to be careful to distinguish between the cases: n is odd, and n is even.

      iv. Show that there are exactly 2^{n-1} subsets whose *size* is an *even* number, and exactly 2^{n-1} subsets whose *size* is an *odd* number.
      Hint: expand $(1 - x)^n$ using the binomial theorem and then set $x = 1$.

2. **[Kitten Combinatorics]**
As I was going to St. Ives, I met a man with seven wives. Each wife had seven sacks. Each sack had seven cats. Each cat had seven kittens. The cats are sleepy, so they stay in the sacks where they belong (seven cats per sack). The kittens, however, are playful and mischievous, often switching sacks, or jumping out of the sacks to play by the side of the road. Fortunately, these are highly intelligent kittens; each kitten meows when its name is called, so it is possible for us to find out which kittens are in any given subset. The kittens are named Aaay, Aaby, Aacy, . . ., Dogy, Dohy, and Doiy.

   (a) Just now, three kittens are sleeping in the sacks; all the others are playing beside the road. What's the probability that the sleeping kittens are Bzky, Cody, and Abby?
   (b) All of the kittens are getting sleepy now, so the eight humans from St. Ives start putting them back into the sacks to sleep. Each kitten chooses a sack "at random" (uniform probability). Define n_k to be the number of kittens bedded down in the kth sack, $1 \le k \le 49$. How many different ways are there to put n_1 kittens in the first sack, n_2 kittens in the second sack, and so on for each sack up through n_{49} kittens in the 49th sack?
   (c) Five kittens wake before the others, and jump out of their sacks to play on the grass. What is the probability that these five kittens have the same mother?
   (d) **Optional:** How many were going to St. Ives?

3. **[Integer Solutions of Equations]**
Global climate change has turned the weather at the South Pole all topsy-turvy; the penguins have no idea which days will be hot, and which cold. Fortunately, they know from a traveling salesman that there will be exactly four cold months, and eight warm months, so they can plan accordingly.

   (a) How many different calendars are compatible with the traveling salesman's prediction? That is, how many different ways are there to divide the year into cold months vs. warm months?
   (b) Of the different calendars you counted in part (a), how many calendars have no consecutive cold months (i.e., there is at least one warm month between every consecutive pair of cold months)?

4. **[Every Probability Class Needs a Question About Poker]**
In five-card stud, the dealer gives each player five cards. Assume that the deck contains only 52 cards: Ace-2-3-. . .-10-J-Q-K, in each of four different suits (hearts, diamonds, clubs, spades). What is the probability that any given five-card hand contains a four-card straight flush (a sequence of four consecutive cards of the same suit, e.g., 4-5-6-7 of spades, or 9-10-J-Q of hearts?) Ace can count either high or low; that is, A-2-3-4 of clubs and J-Q-K-A of diamonds are both straight flushes.

5. **[Drill in working with subsets]**
Find $P(A \cup (B^c \cup C^c)^c)$ in each of the following four cases:

   (a) A, B, and C are disjoint (mutually exclusive) events and $P(A) = 1/3$.
   (b) $P(A) = 2P(B \cap C) = 4P(A \cap B \cap C) = 1/2$.
   (c) $P(A) = 1/2$, $P(B \cap C) = 1/3$, and $P(A \cap C) = 0$.
   (d) $P(A^c \cap (B^c \cup C^c)) = 0.6$.