ECE 313: Problem Set 0
Calculus Review

Due:	*Friday* January 22 at 4 p.m.
Reading:	*Ross* Chapter 1.1–1.4, Chapter 2.1–2.5.
Noncredit Exercises:	Chap. 1: Probs. 1-5,7,9.
	Theoretical exercises 4,8,13. Self-test probs. 1-15.
	Chap. 2: Probs. 3,4,9,10, 11-14.
	Theoretical exerecises 1-3,6,7,10,11,12,16,19,20.
	Self-test probs. 1-8.

About the problems below: Calculus, a prerequisite of this course, will be used mainly in the second half of the semester. The parts of calculus primarily needed are integration and differentiation, Taylor series, L'Hôpital's Rule, integration by parts, and double integrals (setting up limits of integration and change of variables such as in rectangular to polar coordinates). This problem set will help you review these topics, and identify areas in which you may need additional review.

1. [**Geometric, MacLaurin, and Taylor series; L'Hôpital's rule**]

   (a) Prove that $1 + x + x^2 + \cdots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$ for all $x \neq 1$ and all integers $n \geq 1$.

   (b) Continuing to assume that n is a positive integer, what is the value of the sum $1+x+x^2+\cdots+x^{n-1}$ when $x = 1$? Does your answer equal $\lim_{x\to 1} \dfrac{1 - x^n}{1 - x}$?

   (c) Assuming that $|x| < 1$, find the sum of the series $1 + x + x^2 + \cdots$.
   Hint: it is the limit of the finite sum $1 + x + x^2 + \cdots + x^{n-1}$ as $n \to \infty$.

   (d) Prove that
   $$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$
   using L'Hôpital's rule.

   (e) Prove that
   $$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$
   without using L'Hôpital's rule; instead, use the MacLaurin series (Taylor series in the vicinity of 0) for $\sin(x)$.

   (f) What is $d(\sin(x)/x)/dx$ at $x = 0$?

2. [**The binomial theorem**]

   (a) For positive integers n and k, compute the k-th derivative of $f(x) = (1+x)^n$ using the chain rule, that is, *without* multiplying out the terms to get a polynomial in x (as in $(1+x)^2 = 1+2x+x^2$). Use these derivatives to find the MacLaurin series (Taylor series in the vicinity of 0) for $(1+x)^n$.

   (b) According to the textbook (Equation 4.2 in Chapter 1 with $y = 1$),
   $$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$
   Does your answer to part (a) match this result? If so, congratulations! You have just proved the binomial theorem for positive integer exponents.

   (c) Now consider the function $g(x) = (1-x)^{-n}$ where n is a positive integer. Does the MacLaurin series for $g(x)$ contain terms of degree $> n$? If so, what is the term for degree $n+1$? If not, what is the highest degree term?

(d) Use the result of part (c) to write down the MacLaurin series for $(1-x)^{-1}$ and $(1-x)^{-2}$. These results together with the one of parts (a) and (b) will be needed so often in ECE 313 that it is recommended that you memorize them.

(e) Find the MacLaurin series for $(1+x)^a$ where a is a real number and not necessarily an integer.

3. [**Extrema of functions**]

(a) Find all the extreme values (maxima and minima) of $x^{25}(1.00001)^{-x}$ in the interval $(0,\infty)$.

(b) Find the maximum value and the minimum value of $f(x) = \exp(-|x|)$ on the interval $[-1,2]$.

4. [**Some definite integrals**]
Find the values of the following **definite** integrals:

(a) $\displaystyle\int_{-2}^{1} |x|\,dx$; (b) $\displaystyle\int_{0}^{1} x(1-x^2)^{11}\,dx$; (c) $\displaystyle\int_{0}^{1} x^2\exp(-x)\,dx$; (d) $\displaystyle\int_{-10}^{10} x^3\exp(-x^2/2)\,dx$.

5. [**Derivatives and integrals**]
Let $\dfrac{d}{dx}f(x) = g(x),\ -\infty < x < \infty$ and let C denote an arbitrary constant. Which of the following statements is true for all x?

(a) $\frac{d}{dx}f(-x) = -g(-x)$ (b) $\frac{d}{dx}f(x^2/2) = xg(x^2/2)$ (c) $\frac{d}{dx}\exp\left(f(x^2)\right) = g(x^2)\exp\left(f(x^2)\right)$

(d) $\int g(-x)\,dx = f(-x) + C$ (e) $\int g(x^2/2)\,dx = \frac{f(x^2/2)}{x} + C$ (f) $\int \frac{g(x)}{f(x)}\,dx = \ln(f(x)) + C$

6. [**Double integrals**]
Evaluate the following **definite** two-dimensional integrals over the specified domains of integration:

(a) $f(x,y) = \min(x,y)$, over the region $\{(x,y) : 0 \le x \le 2,\ 0 \le y \le 1\}$.

(b) $f(x,y) = \exp(-\frac{1}{2}(x^2+y^2))$ over the region $\{(x,y) : x^2+y^2 > 4\}$.
Hint: change to polar coordinates; then, before tackling the resulting integral, compute the derivative of $\exp(-r^2/2)$ and stare at it for a while!