University of Illinois Spring 2010

ECE 313: Problem Set 11: Solutions

Hazard Rate; Decision Making; Joint Distributions of Random Variables

1. (a) $\{Y > T\} = \{X_1 > T\} \cup \{X_2 > T\}$. By independence,
$P\{Y > T\} = P\{X_1 > T\} + P\{X_2 > T\} - P\{X_1 > T\}P\{X_2 > T\}$.

(b) Because $P\{X_i > T\} = e^{-\lambda T}$, $P\{Y > T\} = e^{-\lambda T} + e^{-\lambda T} - \left(e^{-\lambda T}\right)^2 = 2e^{-\lambda T} - e^{-2\lambda T}$

i. $E[Y] = \int_0^\infty P\{Y > T\}\,dT = \int_0^\infty \left(2e^{-\lambda T} - e^{-2\lambda T}\right)dT = \frac{3}{2\lambda}$

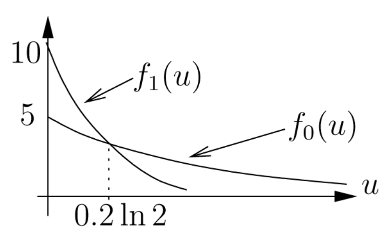
ii. Let $p = e^{-\lambda T}$. Then, $P\{Y > T\} = 2p - p^2 = 1/2$ which has only one solution, $p = 1 - 1/\sqrt{2}$ in the range $0 \le p \le 1$. Thus, the median value of Y is the solution T to $p = e^{-\lambda T} = 1 - 1/\sqrt{2}$, that is, $T = \lambda^{-1}\ln\left(2+\sqrt{2}\right)$.

iii. $f_Y(T) = -\frac{d}{dT}P\{Y > T\} = 2\lambda e^{-\lambda T}\left(1 - e^{-\lambda T}\right)$ for $T > 0$. Therefore,

$$h(T) = \frac{f_Y(T)}{P\{Y > T\}} = \frac{2\lambda e^{-\lambda T}\left(1 - e^{-\lambda T}\right)}{2e^{-\lambda T} - e^{-2\lambda T}} = 2\lambda\frac{\left(1 - e^{-\lambda T}\right)}{2 - e^{-\lambda T}}$$ which goes to zero when T goes to zero.

(c) The dual-component system has a longer MTBF ($1.5\lambda^{-1}$ compared to λ^{-1}), longer half-life $\lambda^{-1}\ln\left(2+\sqrt{2}\right)$ compared to $\lambda^{-1}\ln(2)$. and smaller hazard rate than the single component system.

2. (a)



(b) $\Lambda(u) = \frac{f_1(u)}{f_o(u)} = \frac{10e^{-10u}}{5e^{-5u}} = 2e^{-5u}$ Note that $\Lambda(u) > 1$ for $u < 0.2 \ln 2$. Thus, the likelihood ration test is equivalent to deciding in favor of H_1 if the observed value of X is smaller than the threshold 0.2 ln 2.

(c) $P_{FA} = \int_{\Gamma_1} f_o(u)\,du = \int_0^{0.2\ln 2} 5e^{-5u}du = \frac{1}{2}$ $\qquad P_{MD} = \int_{\Gamma_o} f_1(u)\,du = \int_{0.2\ln 2}^{\infty} 10e^{-10u}du = \frac{1}{4}$

(d) $\Lambda(u) = 2e^{-5u} > \frac{\pi_o}{\pi_1}$ for $u < 0.2\ln\left(\frac{2\pi_1}{\pi_o}\right) = 0.2\ln 2 + 0.2\ln\left(\frac{\pi_1}{\pi_o}\right) = \xi$. Thus, the minimum-error-probability decision rule is equivalent to deciding in favor of H_1 if the observed value of X is smaller than ξ. That is, $\xi < 0$ if $\pi_o > 2\pi_1$ … if $\pi_o > 2/3$.

(e) If $\pi_o = 1/3$, then $\xi < 0.2\ln 4$.

$$P_{FA} = \int_0^{0.2\ln 4} 5e^{-5u}du = \frac{3}{4}$$

$$P_{MD} = \int_{0.2\ln 4}^{\infty} 10e^{-10u}du = \frac{1}{16}$$

The average error probability is $P(E) = \frac{1}{3}P_{FA} + \frac{2}{3}P_{MD} = \frac{7}{24}$. Note that because $\pi_o < \pi_1$, the Bayesian decision rule allows P_{FA} to increase in return for a decrease in P_{MD} because the latter is weighted more heavily.

(f) If the decision rule always decides H_1 is the true hypothesis it makes errors if and only if H_0 is the true hypothesis. Hence, $P(E) = \pi_o$.

(g) When $\pi_o > 2/3$, the threshold ξ is less than 0. Because X takes on nonnegative values, it is always larger than the threshold, and hence the decision is always H_0. The average error probability is π_1, and because this is the minimum-error-probability rule, we cannot do any better than this. Note that $\pi_1 < 1/3$.

When $\pi_o > 2/3$, it follows that $\pi_o > 2\pi_1$. The average probability of error for the maximum-likelihood rule is $\pi_o(1/2) + \pi_1(1/4) > 2\pi_1(1/2) + \pi_1(1/3) = 1.25\pi_1$.

3. (a)

u:	0	1	3	5	Row Sum
v					
4	0	1/12	1/6	1/12	1/3
3	1/6	1/12	0	1/12	1/3
-1	1/12	1/6	1/12	0	1/3
Column Sum	1/4	1/3	1/4	1/6	1

(b) No; $p_{X,Y}(u,v) \neq p_X(u)p_Y(v)$. Check a few to see.

(c) $P\{X \le Y\} = p_{X,Y}(0,3) + p_{X,Y}(0,4) + p_{X,Y}(1,3) + p_{X,Y}(1,4) + p_{X,Y}(3,4) = 1/2$

$P\{X + Y \le 4\} = p_X(0) + p_{X,Y}(1,3) + p_{X,Y}(1,-1) + p_{X,Y}(3,-1) + p_{X,Y}(5,-1) = 7/12$

(d) $p_{X|Y}(u\,|\,3) = \frac{p_{X,Y}(u,3)}{p_Y(3)} = \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ for u = 0,1,5 respectively.

$E[X\,|\,Y = 3] = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 5 \times \frac{1}{4} = \frac{3}{2}$

$$\operatorname{var}(X \mid Y=3)=E\left[X^2 \mid Y=3\right]-\left(E[X \mid Y=3]\right)^2=1\times\frac{1}{4}+5^2\times\frac{1}{4}-\left(\frac{3}{2}\right)^2=\frac{17}{4}$$

4. (a) For any given X = n, Y can take the integer multiple values of n with non-zero probability. Therefore, conditioned on X = n, $p_{Y|X}(y \mid X = n) = P\{Z = y/n\} = (1-q)^{y/n-1}q$, for all y= kn, with k included in N. That is,

$$P_{Y|X}(y \mid n)=\begin{cases} q(1-q)^{y/n}, & y/n \in N \\ 0, & otherwise \end{cases}$$

Using this, we get $p_Y(y)=\sum x=1^y p_{Y|X}(y \mid x)\, p_X(x)=\sum pq(1-q)^{y/n-1}(1-p)^{x-1}$.

(b) Because we are given X = n, we have E[Y | X = n] = E[ZX | X = n] = E[nZ | X = n] = nE[Z | X = n]. Because X and Z are independent, E[Z | X = n] = E[Z] = 1/q. Therefore, E[Y | X = n] = n/q. And, $E[Y^2 \mid X = n] = n^2E[Z^2 \mid X = n] = n^2E[Z^2]$. Finally, var(Y | X = n) = $E[Y^2 \mid X = n] - (E[Y \mid X = n])^2 = n^2 (E[Z^2] - (E[Z])^2) = n^2\operatorname{var}(Z)$. var(Y | X = n) = $n^2(1-q)/q^2$.

5. P{2Y < X} = P{Y < X/2} = $\int_{u=0}^{1}\int_{v=0}^{u/2}(u+v)\,dv\,du=\int_{u=0}^{1}\left(\frac{u^2}{2}+\frac{u^2}{8}\right)du=\frac{5}{24}$

6. P{Z ≤ a} = P{Y/X ≤ a} = P{Y ≤ aX} = $\int_{u=0}^{\infty}\int_{v=0}^{au}e^{-u}\,du=a$

7. (a) $1=\int_{x=0}^{2}\int_{y=0}^{2-x}C_o(x+y)\,dy\,dx=(8/3)C_o$. $C_o = 3/8$.

(b) Z = min{2X,Y}. Then $F_Z(t)$ = P[min{2X,Y} ≤ t] = 1 – P[min{2X,Y} > t] = 1 – P[2X > t, Y > t]. P[2X > t, Y > t] is nonzero if 0 ≤ t ≤ 2 and 2-t/2 ≥ t; t ≤ 4/3. Therefore, for all 0 ≤ t ≤ 4/3, we have $F_Z(t)=1-\int_{x=t/2}^{2-t}\int_{y=t}^{2-x}C_o(x+y)\,dy\,dx=9t/8-27t^3/128$. Differentiating the cdf yields $f_Zt=9/8-81t^2/128$ for 0 ≤ t ≤ 4/3. That is,

$$f_Z(z)=\begin{cases} 9/8-81z^2/128, & 0\le z\le 4/3 \\ 0, & otherwise \end{cases}$$

(c) W = XY. Note that, W can be 1 at most. The reason is, if you have two non-negative numbers whose sume is upper bounded, the pair that maximizes their product is the pair for which each number is equal to the half of the upper bound. In our case, the pair is X + Y ≤ 2 and hence W is maximized for the choice X = Y = 1. For any 0 ≤ t ≤ 1, we have $F_W(t)$ = P[XY ≤ t] = P[Y ≤ t/X] = 1 – P[Y > t/X]. The region of interest is the strip lying between X+ Y = 2 line and the hyperbola defined by Y = t/X. Assume they intersect at points u_o and u_1, and they are functions of t. Solving for X + t/X = 2, we get $u_o=1-\sqrt{1-t}$ and $u_1=1+\sqrt{1-t}$. It is necessary to have t ≤ 1 in order to have the real roots and this is another justification for the support set of W. We have $F_W(t) = 1 - \int_{x=u_o}^{u_1} \int_{y=t/x}^{2-x} C_o(x+y)\,dydx = 1 - C_o\left[2x - x^3/6 - tx + t^2/2x\right] =$ 1- (1-t)$^{3/2}$. That is,

$$f_W(t) = \begin{cases} \frac{3}{2}\sqrt{1-w}, & 0 \le w \le 1 \\ 0, & otherwise \end{cases}$$