University of Illinois Spring 2010

ECE 313: Hour Exam II

Tuesday April 13, 2010
7:00 p.m. — 8:00 p.m.

1. **[25 points]** Over the years that an absent-minded weatherman has been traveling, he has taken spring breaks in either Chicago or Denver. When the weatherman went on spring break this year, he could not remember where he was, but he knew that it must be either Chicago or Denver. Being a weatherman, he knew the following likelihoods of the weather in Chicago and Denver during spring break:

	Sunny	Rain	Snow
Chicago	0.50	0.20	0.30
Denver	0.20	0.30	0.50

The weatherman would like to use this information to make a decision about his current location.

(a) **[5 points]** How many possible decision rules exist for this problem?
Solution: (# of rows) raised to the power of (# of columns) $= 2^3 = 8$

(b) **[5 points]** Determine the maximum likelihood decision rule.
Solution: For any given observed weather condition, the ML rule chooses the city that produces that weather with maximum likelihood:

$$\text{Sunny} \Rightarrow \text{Chicago}$$
$$\text{Rain} \Rightarrow \text{Denver}$$
$$\text{Snow} \Rightarrow \text{Denver}$$

(c) **[5 points]**Referring to his calendar, the weatherman estimated that the prior probability of his being in Chicago on any given spring break is 0.70. He thus concludes that the prior probability that he is in Denver is 0.30. Using this information, fill in the following "Joint probability Matrix for weather in Chicago and Denver:"
Solution:

	Sunny	Rain	Snow
Chicago	0.35	0.14	0.21
Denver	0.06	0.09	0.15

(d) **[5 points]** Assuming the same prior probabilities as in Part (c), determine the probability of error associated with the maximum likelihood decision rule.
Solution: P(error) = 0.06 + 0.14 + 0.21 = 0.41

(e) **[5 points]** Assuming the same prior probabilities as in Part (c), determine the minimum-probability-of-error decision rule (also known as the Bayes decision rule).
Solution: The Bayes decision rule, for any given observed weather condition, chooses the city with maximum joint probability of (weather,city). In this case, the Bayes decision rule is to always choose Chicago.

2. **[25 points]** Consider a random variable X for which

$$P(X > u) = \begin{cases} 1 & u \leq 0 \\ 1 - C \sin u & 0 \leq u < a \\ 0 & a \leq u \end{cases}$$

where u and a are both measured in radians.

(a) **[5 points]** What is the maximum value of a, a_{max}, for which $P(X > u)$ can be a valid complementary CDF?

Solution: A complementary CDF contains no negative probabilities nor can it increase. In order to avoid both such scenarios, we must have $a \le a_{max}$, where $a_{max} = \pi/2$.

(b) **[5 points]** Suppose that $0 < a < a_{max}$. For what value of C is $P(X > u)$ the valid complementary CDF of a continuous random variable?

Solution: If $C > 1/\sin a$, then the CDF is invalid (contains negative probabilities). If $C < 1/sina$, then the CDF contains discontinuities, meaning that X is a mixed RV, not a continuous RV. The CDF is valid and continuous only if $C = 1/\sin a$.

(c) **[5 points]** Calculate the PDF $f_X(u)$ in terms of C and a.

Solution: The PDF is

$$f_X(u) = \begin{cases} C\cos u & 0 < u < a \\ \text{any value} & u = 0 \quad \text{or} \quad u = a \\ 0 & \text{otherwise} \end{cases}$$

(d) **[10 points]** Consider the random variable $Y = 2\cos X$. Find $E[Y]$ in terms of C and a.

Solution: The expectation is

$$E[Y] = \int_0^a 2C\cos^2 u du = C\int_0^a (1 + \cos(2u))\, du = C\left(a + \frac{\sin(2a)}{2}\right)$$

3. **[25 points]** The Klingons have dropped a jamming satellite into low Earth orbit. The satellite emits jamming pulses in a Poisson fashion, at an average rate of λ jamming pulses per minute.

(a) **[6 points]** What's the probability that there will be only three jamming pulses in any given one-hour period?

Solution: The number of pulses per 60 minutes is a Poisson random variable, with mean 60λ, therefore

$$p_X(3) = \frac{(60\lambda)^3}{3!}e^{-60\lambda}$$

(b) **[6 points]** Galactic Federation HQ needs to send a five-minute message from Earth to Arcturus. What's the probability that there are no jamming pulses in any given five-minute period?

Solution: The number of pulses per 5 minutes is a Poisson random variable, with mean 5λ, therefore

$$p_X(0) = e^{-5\lambda}$$

(c) **[6 points]** Suppose that the five-minute period in part (b) is chosen using the following strategy. HQ listens for jamming pulses. After one jamming pulse has been recorded, HQ waits for exactly 90 seconds, then begins sending their message.

If the five-minute period in part (b) were guaranteed to begin exactly 90 seconds after a jamming pulse, would that change your answer to part (b)?

Solution: No. A Poisson process is memoryless: knowing when the last pulse occurred tells you nothing about when the next pulse will occur. Put another way: non-overlapping time intervals are independent.

(d) **[7 points]** By careful measurements, Galactic HQ deduces that the expected length of time between jamming pulses is one quarter of an hour. What is the numerical value of λ?

Solution: The expected length of time between jamming pulses is an exponential random variable, whose expectation is

$$E[Y] = 15 \text{ minutes} = \frac{1}{\lambda}$$

Therefore $\lambda = 1/15$.

4. **[25 points]** Let X be a standard Gaussian random variable (its CDF is $\Phi(s)$). Let $Z = |X|$, and let $Y = e^Z$, where $|X|$ stands for the absolute value of X.

(a) **[10 points]** Find the CDF, $F_Z(u)$, and its PDF, $f_Z(u)$. The CDF should be in terms of $\Phi(u)$; the PDF should not include $\Phi(u)$.

Solution: $Z = |X|$ is a standard one-sided Gaussian RV with pdf $f_Z(u) = \sqrt{\frac{2}{\pi}} \exp(-u^2/2),\ u \geq 0$, and CDF $2\Phi(u) - 1$, $u \geq 0$.

(b) **[15 points]** Find the CDF, $F_Y(s)$, and PDF, $f_Z(s)$, of the random variable $Y = e^Z$. You may express your answer in terms of $F_Z(u)$ and $f_Z(u)$.

Solution: For $s \geq 1$, one has

$$F_Y(s) = P\{Y \leq s\} = P\{\exp Z \leq s\} = P\{Z \leq \log s\} = F_Z(\log s) \tag{1}$$

For $s < 1$, $F_Y(s) = 0$. By taking the derivative of $F_Y(s)$ one obtains that the pdf of Y equals $\frac{1}{s} f_Z(s)$.