

ECE 313: Problem Set 13

Joint Gaussian random variables, MMSE, limit theorems

Due: Wednesday May 6 at 4 p.m.

Reading: Ross Chapter 8

1. [Jointly Distributed Random Variables]

Let the random variables X and Y be such that $E[X] = 1$, $E[Y] = 4$, $Var(X) = 4$, $Var(Y) = 9$, and $\rho = 0.1$. Let $W = 3X + Y + 2$.

- Find $E[W]$ and $Var(W)$.
- If X and Y are jointly Gaussian random variables, what is $P\{W > 0\}$?

2. [Coordinate Transformations for Jointly Distributed Gaussian Random Variables]

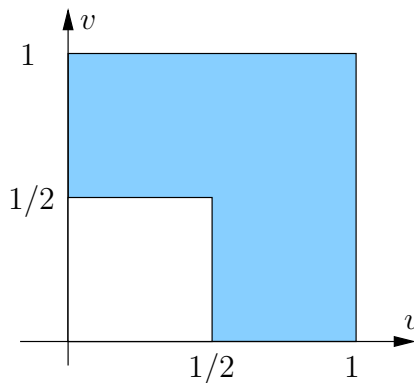
Let X and Y be jointly Gaussian zero-mean random variables, with variances σ_X^2 and σ_Y^2 (respectively) and with correlation coefficient ρ . Find the angle θ such that the random variables $Z = X \cos \theta + Y \sin \theta$ and $W = -X \sin \theta + Y \cos \theta$ are *independent* Gaussian random variables. Note that this coordinate transformation corresponds to a rotation of the axes by θ . Thus, in this problem you will show how to transform two dependent Gaussian random variables to independent Gaussian random variables by a simple rotation.

3. [Linear Minimum Mean Square Error Estimation]

Suppose that the value of Y is to be estimated in terms of the uncorrelated random variables X_1 and X_2 by the linear predictor $g(X_1, X_2) = a + bX_1 + cX_2$. Determine the values for a , b and c that minimize $E[(Y - (a + bX_1 + cX_2))^2]$. Express your answer in terms of the means and variances of X_1 and X_2 and the covariances $Cov(X, Y_1)$ and $Cov(X, Y_2)$.

4. [Minimum Mean Square Error Estimation]

This problem is a continuation of Problem 8 from Problem Set 11. The random point (X, Y) is uniformly distributed on the shaded region shown.



- Given that the value of X is u , the (conditional) minimum-mean-square-error estimate of Y is $E[Y|X = u]$, the conditional mean of Y , and the (conditional minimum) mean-square error achieved is $VAR(Y|X = u)$.

Use your answers to Problem 8 from Problem Set 11 to sketch graphs of $E[Y|X = u]$ and $VAR(Y|X = u)$ as functions of u for $0 < u < 1$.

- Since $VAR(Y|X = u)$ depends on the value of X , it is a *function* of X . Use LOTUS to compute $E[VAR(Y|X = u)]$, the expected value of this function. This is the (unconditional) mean-square error that we achieve when we estimate Y as $E[Y|X = u]$, and *no* other estimate can have smaller mean-square error than this.

- (c) It can be easily shown that $\text{cov}(X, Y) = -\frac{1}{36}$ and $\rho_{X,Y} = -\frac{4}{11}$. Now, the minimum-mean-square-error *linear* estimate of Y given that the value of X is u is

$$\hat{Y} = E[X] + \rho_{X,Y} \sqrt{\text{VAR}(Y)/\text{VAR}(X)}(u - E[X]) = -\frac{4}{11}u + \frac{35}{44},$$

and the (unconditional) mean-square error of this linear estimate is $\text{VAR}(Y)(1 - \rho_{X,Y}^2)$.

Sketch $\hat{Y}$ as a function of u on the same graph that you used in part (e) and compare $\hat{Y}$ to the nonlinear (optimum) estimate $E[Y|X = u]$. Which is larger, $\hat{Y}$ when $X = u = 0$ or $E[Y|X = 0]$? Which is larger, $\hat{Y}$ when $X = u = 1$ or $E[Y|X = 1]$?

Is $E[\text{VAR}(Y|X = u)] \leq \text{VAR}(Y)(1 - \rho_{X,Y}^2)$ as it should be?

5. **[Conditional Expectation]**

Problem 7.73, page 379 of Ross.

6. **[Functions of i.i.d. Gaussian Random Variables]**

Let X and Y be jointly continuous, independent random variables with $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \sigma^2)$. (NOTE: random variables that are independent and share the same pdf are called independent, identically distributed, or i.i.d. random variables). Determine the pdf of $Z = X^2 + Y^2$.

7. **[The Central Limit Theorem]**

Suppose that each of 100 real numbers are rounded to the nearest integer and then added. Assume that the individual roundoff errors are independent and uniformly distributed over the interval $[-0.5, 0.5]$. Using the normal approximation suggested by the central limit theorem, find the probability that the absolute value of the sum of the errors is greater than 5.0.

8. **[Limit Theorems]**

A single fair die is rolled repeatedly. Let X_i denote the number appearing on the i^{th} roll. Assume that the X_i are independent, and let $S_n = X_1 + X_2 + \cdots + X_n$.

- Use Markov's inequality to find an upper bound on $P\{S_{100} \geq 400\}$.
- Use Chebyshev's inequality to find an upper bound on $P\{S_{100} \geq 400\}$. (Hint: a one-sided tail probability can be bounded by a two-sided tail probability.)
- Compute the approximation to $P\{S_{100} \geq 400\}$ suggested by the central limit theorem.