

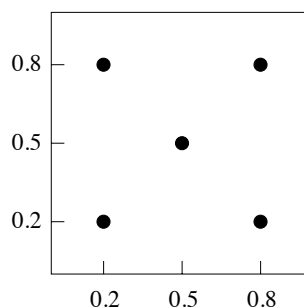
ECE 313: Problem Set 12

Functions of random variables, conditional pdfs, covariance

Due:	Wednesday April 29 at 4 p.m..
Reading:	<i>Ross</i> Chapters 6 and 7.
Noncredit Exercises:	<i>Ross</i> , Chapter 6, problems 26, 28-30, 41-43, 51, 54, and theoretical exercises 8, 14, 22, 23, 33, and Chapter 7, problems 1, 16, 29, 34, 36 and theoretical exercises 1, 2, 17, 22, 23, 40.

1. [Analysis of a vector quantizer]

A vector quantizer is a function, g , mapping an input region in multiple dimensions to a finite subset of the region. For example, let g be the minimum distance mapping for the five mapping points in the unit square shown.



Specifically, for (x, y) in the unit square, the output $g(x, y)$ is the mapping point (w, z) closest to (x, y) . For example, $g(0.3, 0.3) = (0.2, 0.2)$ and $g(0.4, 0.6) = (0.5, 0.5)$.

- Find the partition of the unit square into five regions, corresponding to the five possible values of g . For example, one of the regions is the set of all (x, y) closer to $(0.2, 0.2)$ than to any of the other four mapping points.
- Suppose that (X, Y) is uniformly distributed over the unit square, and $(W, Z) = g(X, Y)$. Determine the joint pmf of (W, Z) .

2. [Functions of independent exponential random variables]

Let X_1 and X_2 be independent random variables, with X_i being exponentially distributed with parameter λ_i for $i \in \{1, 2\}$.

- Find the pdf of $Z = \min\{X_1, X_2\}$.
- Find the pdf of $R = \frac{X_1}{X_2}$.

3. **[Uniform distribution on a rotated square]**

Random variables X and Y have a uniform joint density on the square bounded with corners at the points: $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.

- (a) Calculate the marginal pdfs of X and Y . Are X and Y independent?
- (b) Compute $E[X]$ and $\text{Var}(X)$.
- (c) Calculate the pdf of the random variable $A = X + Y$.
- (d) Calculate the pdf of the random variable $C = X/Y$.

4. **[Using a joint density]**

Suppose X and Y are jointly continuous random variables distributed over the unit square with the joint pdf given by

$$f_{X,Y}(u, v) = \begin{cases} \frac{3u^2}{2} + 2uv & u, v \in [0, 1] \\ 0 & \text{else} \end{cases}$$

- (a) Are X and Y independent? Briefly justify your answer.
- (b) Calculate $E[X]$.
- (c) Calculate the correlation, $E[XY]$.
- (d) Calculate the pdf, $f_Y(v)$, of Y . Be sure to specify it for $-\infty < v < \infty$.
- (e) Calculate the conditional density $f_{X|Y}(u|v)$. Be sure to indicate what values of v it is well-defined for, and for such v , specify it for $-\infty < u < \infty$.

5. **[Some moments for a random rectangle]**

Let $A = XY$ denote the area and $L = 2(X + Y)$ the length of the perimeter, of a rectangle with length X and height Y , such that X and Y are independent, and uniformly distributed on the interval $[0, 1]$.

- (a) Find $E[A]$ and $E[L]$.
- (b) Find $\text{Var}(A)$. (Hint: Find $E[A^2]$ first.)
- (c) Find $\text{Var}(L)$.
- (d) Find $\text{Cov}(A, L)$. (Hint: Find $E[AL]$ first.)
- (e) Find the correlation coefficient, $\rho_{A,L}$. (Hint: Should be less than, but fairly close to, one. Why?)

6. **[Working with independent Gaussian random variables]**

Let X and Y be independent, $N(0, 1)$ random variables.

- (a) Find $\text{Cov}(3X + 2Y, X + 5Y + 10)$.
- (b) Express $P\{X + 4Y \geq 2\}$ in terms of the Q function defined by $Q(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$.
- (c) Express $P\{(X - Y)^2 > 9\}$ in terms of the Q function. (Hint: For what values of $X - Y$ is the event true?)