

ECE 313: Problem Set 13: Solutions

Joint Gaussian random variables, MMSE, limit theorems

1. [Jointly Distributed Random Variables]

(a) $E[W] = E[3X + Y + 2] = 3E[X] + E[Y] + 2 = 9$.
 $Var(W) = Var(3X + Y + 2) = 3^2 \cdot Var(X) + Var(Y) + 2 \cdot 3 \cdot 1 \cdot Cov(X, Y) = 9 \cdot 4 + 9 + 6 \cdot 2 \cdot 3 \cdot 0.1 = 48.6$.

(b) $P\{W > 0\} = 1 - \Phi\left(\frac{0-9}{\sqrt{48.6}}\right) = 1 - \Phi\left(-\frac{9}{\sqrt{48.6}}\right) = \Phi\left(\frac{9}{\sqrt{48.6}}\right)$

2. [Coordinate Transformations for Jointly Distributed Gaussian Random Variables]

Z and W are jointly Gaussian random variables and independent if $Cov(Z, W) = 0$.

Since $Cov(Z, W) = E[ZW] - E[Z]E[W] = E[(X \cos \theta + Y \sin \theta)(Y \cos \theta - X \sin \theta)]$

$= E[Y^2] \sin \theta \cos \theta - E[X^2] \sin \theta \cos \theta + E[XY](\cos^2 \theta - \sin^2 \theta)$

$= \sin \theta \cos \theta \cdot (\sigma_y^2 - \sigma_x^2) + (\cos^2 \theta - \sin^2 \theta) \cdot \rho \sigma_y \sigma_x = \frac{1}{2} \cdot \sin 2\theta \cdot (\sigma_y^2 - \sigma_x^2) + \cos 2\theta \cdot \rho \sigma_y \sigma_x$, we get independent

random variables if we choose $\theta = \frac{1}{2} \arctan\left(\frac{2\rho \cdot \sigma_y \sigma_x}{\sigma_x^2 - \sigma_y^2}\right)$. Note that if θ_0 is a solution to this equation,

then $\theta_0 + \pi$ is also a solution, as are $\theta_0 \pm \pi/2$. That is, there are *four* different values of θ in the range $[0, 2\pi)$ that can be used to get independent Gaussian random variables from X and Y . In particular, if $\sigma_1 = \sigma_2$, then θ can take on values $\pm\pi/4$ and $\pm 3\pi/4$.

3. [Linear Minimum Mean Square Error Estimation]

$\frac{dMSE}{da} = 2E[Y - (a + bX_1 + cX_2)(-1)] = 0 \rightarrow a = \mu_Y - b\mu_{X_1} - c\mu_{X_2}$

$\frac{dMSE}{db} = 2E[Y - (a + bX_1 + cX_2)(-X_1)] = 0 \rightarrow bE[X_1^2] = E[X_1Y] - a\mu_{X_1} - c\mu_{X_1}\mu_{X_2}$

$b(Var(X_1) + \mu_{X_1}^2) = Cov(X_1, Y) + E[X_1]E[Y] - a\mu_{X_1} - c\mu_{X_1}\mu_{X_2}$

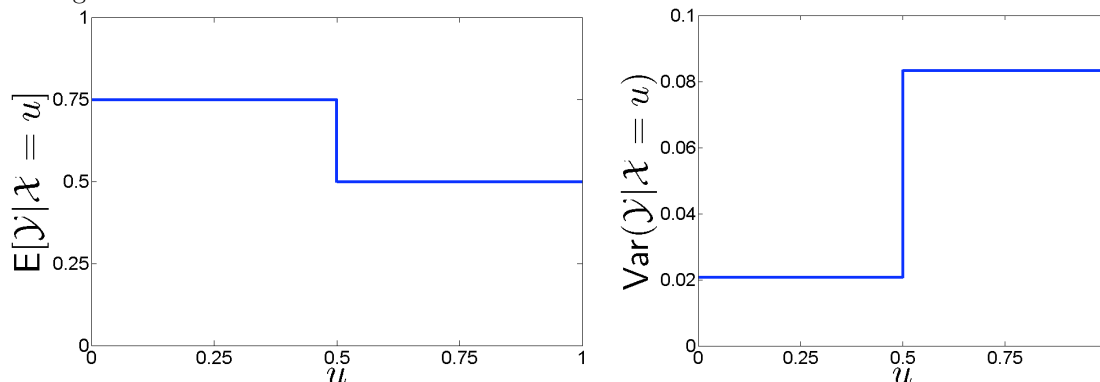
$\frac{dMSE}{dc} = 0 \rightarrow c(Var(X_2) + \mu_{X_2}^2) = Cov(X_2, Y) + E[X_2]E[Y] - a\mu_{X_2} - b\mu_{X_1}\mu_{X_2}$

Plugging in a , $b(Var(X_1) + \mu_{X_1}^2) = Cov(X_1, Y) + E[X_1]E[Y] - (\mu_Y - b\mu_{X_1} - c\mu_{X_2})\mu_{X_1} - c\mu_{X_1}\mu_{X_2}$.

$b(Var(X_1) + \mu_{X_1}^2) = Cov(X_1, Y) + b\mu_{X_1}$ and $b = \frac{Cov(X_1, Y)}{Var(X_1)}$. Similarly, $c = \frac{Cov(X_2, Y)}{Var(X_2)}$ and $a = \mu_Y - \frac{Cov(X_1, Y)\mu_{X_1}}{Var(X_1)} - \frac{Cov(X_2, Y)\mu_{X_2}}{Var(X_2)}$

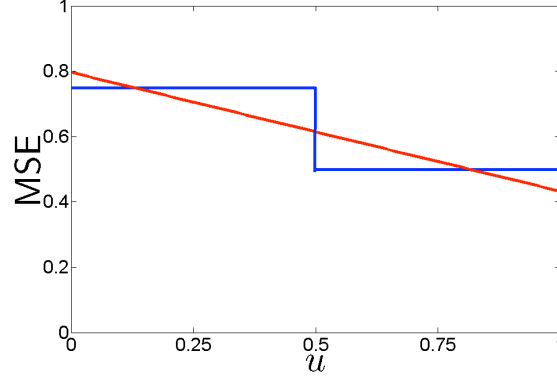
4. [Minimum Mean Square Error Estimation]

(a) See figures below.



(b) MSE of MMSE estimator $= E[\text{var}(Y|X=u)] = \int_0^{1/2} \frac{1}{48} \cdot \frac{2}{3} du + \int_{1/2}^1 \frac{1}{12} \cdot \frac{4}{3} du = \frac{7}{432}$

(c) See figure below.



From the figure, $\hat{\mathbb{Y}}$ is larger than $E[\mathbb{Y}|\mathbb{X} = 0]$, when $\mathbb{X} = u = 0$ and $E[\mathbb{Y}|\mathbb{X} = 1]$ is larger than $\hat{\mathbb{Y}}$ when $\mathbb{X} = u = 1$.

$$\text{var}(\mathbb{Y}) (1 - \rho_{\mathbb{X}, \mathbb{Y}}^2) = \frac{11}{144} \left(1 - \left(\frac{4}{11} \right)^2 \right) = \frac{35}{528} > \frac{7}{432} \text{ as it should be.}$$

5. [Conditional Expectation]

(a) $E[R] = E[E[R|S]] = E[S] = \mu$. (Proposition 5.1)

(b) $\text{Var}(R) = E[\text{Var}(R|S)] + \text{Var}(E[R|S]) = E[1] + \text{Var}(S) = 1 + \sigma^2$. (Proposition 5.2)

(c)

$$\begin{aligned} f_R(r) &= \int_{-\infty}^{\infty} f_{R,S}(r, s) ds = \int_{-\infty}^{\infty} f_{R|S}(r|s) f_S(s) ds = C_1 \int_{-\infty}^{\infty} e^{-\frac{(r-s)^2}{2}} e^{-\frac{(s-\mu)^2}{2\sigma^2}} ds \\ &= C_1 \int_{-\infty}^{\infty} e^{-\frac{\sigma^2(r-s)^2 + (s-\mu)^2}{2\sigma^2}} ds \end{aligned} \quad (1)$$

where C_1 is a constant. At this point, we can observe that we can simplify $-\frac{\sigma^2(r-s)^2 + (s-\mu)^2}{2\sigma^2}$ as is done in the book in example 6b, although noting the additional terms of r , namely

$$f_R(r) = C_1 \left[\int_{-\infty}^{\infty} e^{-\left(\frac{s - \frac{\mu + r\sigma^2}{1 + \sigma^2}}{\frac{\sigma^2}{1 + \sigma^2}} \right)^2} ds \right] e^{ar^2 + br}$$

where a and b are nonzero coefficients. Since we are integrating over a gaussian pdf (well, without the normalizing coefficients) we get a constant (what would be 1 over the normalizing coefficient), resulting in $f_R(r) = C_2 e^{ar^2 + br}$. We can “complete the square” to put this into the traditional form for a gaussian pdf. Because of this result, R is normally distributed, namely $f_R(r) = \frac{1}{\sqrt{2\pi(1+\sigma^2)}} e^{-\frac{(r-\mu)^2}{2(1+\sigma^2)}}$.

(d) $\text{Cov}(R, S) = E[RS] - E[R]E[S] = E[E[RS|S]] - \mu\mu = E[S^2] - \mu^2 = (\text{Var}(S) + E[S]) - \mu^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$.

6. [Functions of i.i.d. Gaussian Random Variables]

$F_Z(z) = \int \int_{(x,y) \in C_z} f_{XY}(x, y) dx dy$ for $z \geq 0 = \frac{1}{2\pi\sigma^2} \int \int_{x^2 + y^2 \leq z} \exp -\frac{1}{2\sigma^2}(x^2 + y^2) dx dy$. Using polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, and $dx dy \rightarrow r dr d\theta$. $F_Z(z) = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r \exp -\frac{1}{2\sigma^2}(r^2) dr$
 $= [1 - \exp -\frac{z}{2\sigma^2}] u(z)$. $f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{2\sigma^2} \exp -\frac{z}{2\sigma^2} u(z)$. Thus Z is an exponential random variable. Equivalently Z is a chi-squared random variable with two degrees of freedom.

7. [The Central Limit Theorem]

The mean of each roundoff error is zero and the variance is $\int_{-0.5}^{0.5} u^2 du = \frac{1}{12}$. Thus, $E[S] = 0$ and $Var(S) = \frac{100}{12} = 8.333$. Thus, $P[|S| \geq 5] = P[|\frac{S}{\sqrt{8.333}}| \geq \frac{5}{\sqrt{8.333}}] \approx 2Q(\frac{5}{\sqrt{8.333}}) = 2Q(1.73) = 2(1 - \Phi(1.732)) = 0.083$.

8. [Limit Theorems]

(a) $E[X_1] = \frac{1+2+3+4+5+6}{6} = 3.5$ so by Markov's inequality, $P[S_{100} \geq 400] \leq \frac{100(3.5)}{400} = \frac{7}{8}$. (Not at all a strong bound.)

(b) $E[X_1^2] = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = 15$ so that $Var(X_i) = 15 - 3.5^2 = 2.75$ Therefore by Chebyshev's inequality $P[S_{100} \geq 400] = P[S_{100} - 350 \geq 50] \leq P[|S_{100} - 350| \geq 50] \leq \frac{100(2.75)}{50^2} = 0.11$
Or, we could use the one-side Chebyshev inequality found on page 418 of Ross. That yields $P[S_{100} \geq 400] = P[S_{100} - 350 \geq 50] \leq \frac{275}{275+2500} = 0.0991$. (Either answer is acceptable.)

(c)

$$P[S_{100} \geq 400] = P\left[\frac{S_{100} - 350}{\sqrt{(2.75)100}} \geq \frac{400 - 350}{\sqrt{(2.75)100}}\right] \approx Q\left(\frac{400 - 350}{\sqrt{(2.75)100}}\right) = Q(3.015) = 1 - \Phi(3.015) \approx 0.0009$$