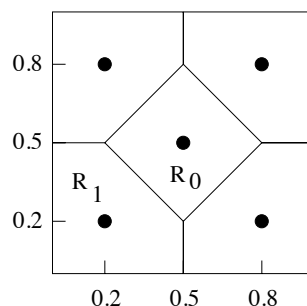


ECE 313: Problem Set 12: Solutions

Functions of random variables, conditional pdfs, covariance

1. [Analysis of a vector quantizer]

- (a) The mapping points induce the so-called Voronoi diagram, as shown.



The square is partitioned into five cells, such that each cell is the set of points in the square closest to one of the mapping points. For example, the line segment separating regions R_0 and R_1 has endpoints $(0.5, 0.2)$ and $(0.2, 0.5)$. This segment is a portion of the line $\{(u, v) : u + v = 0.7\}$, which is equally distant from the mapping points in regions R_0 and R_1 .

- (b) The area of cell R_1 is $(0.5)^2 - (0.5)(0.3)^2 = 0.205$. The other three corner cells have the same area. The center cell, R_0 , thus has area $1 - 4(0.205) = 0.18$. Hence, the joint pmf of (W, Z) is

$$p_{W,Z}(\alpha, \beta) = \begin{cases} 0.205 & \text{if } (\alpha, \beta) = (0.5 \pm 0.3, 0.5 \pm 0.3) \\ 0.18 & \text{if } (\alpha, \beta) = (0.5, 0.5) \end{cases}$$

2. [Functions of independent exponential random variables]

- (a) Z takes values in the positive real line. So let $c \geq 0$.

$$\begin{aligned} P[Z \leq c] &= P[\min\{X_1, X_2\} \leq c] = P[X_1 \leq c \text{ or } X_2 \leq c] \\ &= 1 - P[X_1 > c \text{ and } X_2 > c] = 1 - P[X_1 > c]P[X_2 > c] = 1 - e^{-\lambda_1 c}e^{-\lambda_2 c} = 1 - e^{-(\lambda_1 + \lambda_2)c} \end{aligned}$$

Differentiating yields that

$$f_Z(c) = \begin{cases} (\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)c}, & c \geq 0 \\ 0, & c < 0 \end{cases}$$

That is, Z has the exponential distribution with parameter $\lambda_1 + \lambda_2$. (Note: Recall that the exponential distribution with parameter λ is the distribution with constant failure rate λ . Here, Z can be viewed as the failure time of a system which fails when either of two components fails, where X_1 and X_2 are the failure times of the two components. This problem is an example of the fact that the failure rate of such a system is the sum of the failure rates of the two components.)

- (b) R takes values in the positive real line and by independence the joint pdf of X_1 and X_2 is the product of their individual densities. So for $r \geq 0$,

$$\begin{aligned} P[R \leq r] &= P\left[\frac{X_1}{X_2} \leq r\right] = P[X_1 \leq rX_2] \\ &= \int_0^\infty \int_0^{rx_2} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} dx_1 dx_2 \\ &= \int_0^\infty (1 - e^{-r\lambda_1 x_2}) \lambda_2 e^{-\lambda_2 x_2} dx_2 = 1 - \frac{\lambda_2}{r\lambda_1 + \lambda_2}. \end{aligned}$$

Differentiating yields that

$$f_R(r) = \begin{cases} \frac{\lambda_1 \lambda_2}{(\lambda_1 r + \lambda_2)^2} & r \geq 0 \\ 0, & r < 0 \end{cases}$$

3. [Uniform distribution on a rotated square]

- (a) X and Y are not independent since the footprint of their joint pdf is not rectangular. The area of the square region is 2 so that the pdf of (X, Y) is $\frac{1}{2}$ within the region. Integrating out the v variable of the joint density function yields the triangle shaped pdf:

$$f_X(u) = \begin{cases} \int_{-1-u}^{1-u} \frac{1}{2} dv = 1 + u, & -1 < u \leq 0 \\ \int_{u-1}^{1-u} \frac{1}{2} dv = 1 - u, & 0 < u \leq 1 \\ 0 & \text{else} \end{cases}$$

Equivalently,

$$f_X(u) = \begin{cases} 1 - |u| & -1 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

Since the joint pdf is symmetric in u and v , it follows that the marginal density for Y is exactly the same.

- (b)

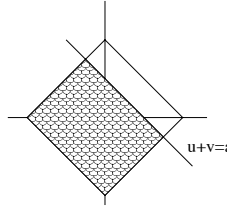
$$E[X] = \int_{-\infty}^{\infty} u f_X(u) du = \int_{-1}^1 u(1 - |u|) du = 0$$

since an odd function integrates to zero.

$$E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u) du = \int_{-1}^1 u^2(1 - |u|) du = 2 \int_0^1 u^2(1 - u) du = 2 \int_0^1 u^2 - u^3 du = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

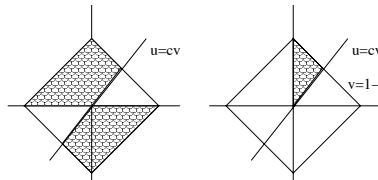
So $\text{Var}(X) = E[X^2] - E[X]^2 = \frac{1}{6}$.

- (c) The range of A is the interval $[-1, 1]$. If a is in this interval, then the event $\{A \leq a\}$ is the same as (X, Y) falling within the halfspace $\{(u, v) : u + v \leq a\}$. The halfspace intersects the rotated square in a rectangle of dimensions $\frac{1+a}{\sqrt{2}} \times \sqrt{2}$. See the figure.



The area of this rectangle is $1 + a$. Since the pdf of (X, Y) is $\frac{1}{2}$ within the rectangle, $P(A \leq a) = \frac{1+a}{2}$. Hence, $X + Y$ is uniformly distributed over the interval $[-1, 1]$.

- (d) The range of C is the whole real line. Our approach will be to first find the CDF and pdf of C within the region $c \geq 0$. So let $c \geq 0$. Then $F_C(c)$ is one half the area of the shaded region in the diagram on the left below, which is 0.5 plus the area of the shaded region in the diagram on the right. View the area as a triangle rotated clockwise a quarter revolution. The area of the triangle



is one half the base times the height. The base is the unit length segment from $(0, 0)$ to $(0, 1)$.

The height is the value of u at which the line $\{u = cv\}$ intersects the edge of the square, given by the equation $v = 1 - u$. Solving for u yields $u = \frac{c}{1+c}$. The area of the shaded triangle is thus $\frac{u}{2} = \frac{c}{2(1+c)}$ so that $F_C(c) = 0.5 + \frac{c}{2(1+c)}$ for $c \geq 0$. Differentiating this with respect to c yields that $f_C(c) = \frac{1}{2(1+c)^2}$ for $c \geq 0$.

We could find the pdf for negative c by a similar method. However, it is less work to notice that (X, Y) and $(-X, Y)$ have the same joint distribution, so that C has the same distribution as $-C$. Thus, f_C is an even function: $f_C(-c) = f_C(c)$. Therefore, $f_C(c) = \frac{1}{2(1+|c|)^2}$ for all real values of c .

4. [Using a joint density]

(a) No, because the joint pdf $f_{X,Y}(u, v)$ cannot be expressed as a function of u times a function of v .

$$(b) E[X] = \int_0^1 \int_0^1 u \left(\frac{3u^2}{2} + 2uv \right) dudv = \int_0^1 \frac{3u^3}{2} du \int_0^1 1dv + \int_0^1 2u^2 du \int_0^1 vdv = \frac{3}{8} + \frac{2}{3} \frac{1}{2} = \frac{17}{24}$$

$$(c) E[XY] = \int_0^1 \int_0^1 uv \left(\frac{3u^2}{2} + 2uv \right) dudv = \int_0^1 \frac{3u^3}{2} du \int_0^1 vdv + \int_0^1 2u^2 du \int_0^1 v^2 dv = \frac{3}{8} \frac{1}{2} + \frac{2}{3} \frac{1}{3} = \frac{59}{144}$$

$$(d) f_Y(v) = \begin{cases} \int_0^1 \left(\frac{3u^2}{2} + 2uv \right) du = \frac{1}{2} + v & v \in [0, 1] \\ 0 & \text{else} \end{cases}$$

$$(e) f_{X|Y}(u|v) \text{ is not defined unless } v \in [0, 1]. \text{ For } v \in [0, 1], f_{X|Y}(u|v) = \begin{cases} \frac{\frac{3u^2}{2} + 2uv}{\frac{1}{2} + v} & u \in [0, 1] \\ 0 & \text{else} \end{cases}$$

5. [Some moments for a random rectangle]

(a)

$$\begin{aligned} E[A] &= E[XY] = E[X]E[Y] \quad \text{because } X \text{ and } Y \text{ are independent} \\ &= \left(\frac{1}{2} \right)^2 = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} E[L] &= 2E[X] + 2E[Y] \quad \text{by linearity of expectation} \\ &= 2 \end{aligned}$$

(b)

$$\begin{aligned} E[A^2] &= E[X^2Y^2] = E[X^2]E[Y^2] \quad \text{because } X^2 \text{ and } Y^2 \text{ are independent} \\ &= \left(\frac{1}{3} \right)^2 = \frac{1}{9} \end{aligned}$$

$$\text{Therefore, } \text{Var}(A) = E[A^2] - E[A]^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}.$$

(c)

$$\begin{aligned} \text{Var}(L) &= \text{Var}(2(X + Y)) = 4\text{Var}(X + Y) \quad \text{variance scales quadratically} \\ &= 4(\text{Var}(X) + \text{Var}(Y)) \quad \text{variance of sum is sum of variances for independent r.v.'s} \\ &= 4 \left(\frac{1}{12} + \frac{1}{12} \right) = \frac{2}{3} \end{aligned}$$

$$(d) E[AL] = E[2XY(X + Y)] = 2E[X^2]E[Y] + 2E[X]E[Y^2] = 2 \cdot \frac{1}{3} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}.$$

Therefore, $\text{Cov}(A, L) = E[AL] - E[A]E[L] = \frac{2}{3} - \frac{1}{4} \cdot 2 = \frac{1}{6}.$

$$(e) \rho_{A,L} = \frac{\text{Cov}(A,L)}{\sqrt{\text{Var}(A)\text{Var}(L)}} = \frac{\frac{1}{6}}{\sqrt{\frac{7}{144} \cdot \frac{2}{3}}} = \sqrt{\frac{6}{7}} \approx 0.92582.$$

6. [Working with independent Gaussian random variables]

(a)

$$\begin{aligned}\text{Cov}(3X + 2Y, X + 5Y + 10) &= \text{Cov}(3X + 2Y, X + 5Y) \\ &= 3\text{Cov}(X, X) + 17\text{Cov}(X, Y) + 10\text{Cov}(Y, Y) \\ &= 3\text{Var}(X) + 0 + 10\text{Var}(Y) = 13.\end{aligned}$$

(b) Let $S = X + 4Y$. Then S is the sum of the independent Gaussian random variables X and $4Y$. Thus, S is a Gaussian random variable. Also, $E[S] = E[X] + 4E[Y] = 0$, and $\text{Var}(S) = \text{Cov}(X + 4Y, X + 4Y) = \text{Var}(X) + 8\text{Cov}(X, Y) + 16\text{Var}(Y) = 1 + 0 + 16 = 17$. So S has the $N(0, 17)$ distribution. Thus, $P\{X + 4Y \geq 2\} = P\{\frac{S}{\sqrt{17}} \geq \frac{2}{\sqrt{17}}\} = Q(\frac{2}{\sqrt{17}})$.

(c) Let $D = X - Y$. Proceeding as in part (a), we see that D is a $N(0, 2)$ random variable. So $P\{(X - Y)^2 > 9\} = P\{D^2 > 9\} = P\{D \geq 3 \text{ or } D \leq -3\} = 2P\{D \geq 3\} = 2P\left\{\frac{D}{\sqrt{2}} \geq \frac{3}{\sqrt{2}}\right\} = 2Q(\frac{3}{\sqrt{2}})$.