

ECE 313: Problem Set 10: Solutions

Functions of a random variable, hazard rates, decision making

1. [System reliability]

- (a) The median lifetime is the value
- T
- that satisfies

$$\frac{1}{2} = P\{X > T\} = \int_T^\infty (bu) \exp\left(-\frac{bu^2}{2}\right) du \Rightarrow -\exp\left(-\frac{bu^2}{2}\right) \Big|_T^\infty = \exp\left(-\frac{bT^2}{2}\right) \Rightarrow T = \sqrt{\frac{2 \ln 2}{b}}$$

Using the substitution $v = bu^2/2$, the mean for a Rayleigh random variable is given by

$$E[X] = \int_0^\infty u(bu) \exp\left(-\frac{bu^2}{2}\right) du = \sqrt{2/b} \int_0^\infty \sqrt{v} e^{-v} dv.$$

But the last integral is just the gamma function $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \sqrt{\pi}/4$, which gives $E[X] = \sqrt{2/b} \sqrt{\pi}/4 = \sqrt{\pi/2b}$. (NOTE: showing that $\Gamma(1/2) = \sqrt{\pi}$ is exercise 20, pg. 239 of Ross.)

The mode is easily found to be at $u = \sqrt{1/b}$. Since $\sqrt{1/b} < \sqrt{2 \ln 2} \sqrt{1/b} < \sqrt{\pi/2} \sqrt{1/b}$, we see that the mode is less than the median is less than the mean.

- (b)

$$0 = \frac{d}{db}(bt) \exp\left(-\frac{bt^2}{2}\right) = t \exp\left(-\frac{bt^2}{2}\right) - \frac{t^2}{2} b t \exp\left(-\frac{bt^2}{2}\right) = t \exp\left(-\frac{bt^2}{2}\right) \left(1 - \frac{t^2}{2} b\right) \Rightarrow b = \frac{2}{t^2}.$$

2. [Functions of a random variable I]

Clearly Y is a continuous random variable, and therefore we will find its pdf (not a pmf). The random variable Y takes all values in the interval $(0, \infty)$. Now, if $v > 0$, then $0 < \exp(-v) < 1$ and $F_Y(v) = P\{Y \leq v\} = P\{-\ln X \leq v\} = P\{X > \exp(-v)\} = 1 - \exp(-v)$, since $X \sim U(0, 1)$. Hence,

$$f_Y(v) = \frac{d}{dv} F_Y(v) = \begin{cases} \exp(-v), & v > 0 \\ 0 & \text{otherwise.} \end{cases}$$

3. [Functions of a random variable II]

- (a) Clearly,
- Y
- is a discrete random variable (it can take only the two values
- $\{-1, 1\}$
-), so we will determine its pmf.

$$p_Y(-1) = P\{Y = -1\} = P\{X + 2 < 0\} = P\{X < -2\} = \frac{1}{2} \int_{-\infty}^{-2} u^{-2} du = -\frac{1}{2} u^{-1} \Big|_{-\infty}^{-2} = \frac{1}{4}$$

$$\begin{aligned} \text{and } p_Y(1) &= P\{Y = 1\} = P\{X + 2 \geq 0\} = P\{X \geq -2\} = \frac{1}{2} \int_{-2}^{-1} u^{-2} du + \frac{1}{2} \int_1^{\infty} u^{-2} du \\ &= -\frac{1}{2} u^{-1} \Big|_{-2}^{-1} + \frac{1}{2} u^{-1} \Big|_1^{\infty} = \frac{1}{2} - \frac{1}{4} + \frac{1}{2} = \frac{3}{4}. \end{aligned}$$

Of course the easier way to determine $p_Y(1)$ is to use $p_Y(1) = 1 - p_Y(-1)$.

- (b)
- Z
- is a continuous random variable, and to find its pdf, we will find the CDF
- $F_Z(v)$
- and differentiate. First, note that
- Z
- cannot be negative regardless of the value of
- X
- , and since
- $|X| > 1$
- , we see that

$Z \geq 2$. Thus, $F_Z(v) = 0$ for all $v < 2$. For $1 \leq |X| \leq 2$, we have $Z = 2|X|$ and thus $2 \leq Z \leq 4$. So, for $2 \leq v \leq 4$ we have

$$\begin{aligned} F_Z(v) &= P\{Z \leq v\} = P\{2|X| \leq v\} = P\{-\frac{v}{2} \leq X \leq \frac{v}{2}\} = \frac{1}{2} \int_{-v/2}^{-1} u^{-2} du + \frac{1}{2} \int_1^{v/2} u^{-2} du \\ &= -\frac{1}{2} u^{-1} \Big|_{-v/2}^{-1} - \frac{1}{2} u^{-1} \Big|_1^{v/2} = \left(\frac{1}{2} - \frac{1}{v}\right) - \left(\frac{1}{v} - \frac{1}{2}\right) = 1 - \frac{2}{v} \end{aligned}$$

Next, for $|X| > 2$ we have $Z = X^2$ and thus, for $v > 4$ we obtain

$$\begin{aligned} F_Z(v) &= P\{Z \leq v\} = P\{X^2 \leq v\} = P\{-\sqrt{v} \leq X \leq \sqrt{v}\} = 2P\{1 \leq X \leq \sqrt{v}\} \\ &= 2 * \frac{1}{2} \int_1^{\sqrt{v}} u^{-2} du = -u^{-1} \Big|_1^{\sqrt{v}} = 1 - \frac{1}{\sqrt{v}} \end{aligned}$$

Putting this all together, we obtain

$$F_Z(v) = \begin{cases} 0, & v < 2 \\ 1 - \frac{2}{v}, & 2 \leq v \leq 4, \text{ and upon differentiating, we get } f_Z(v) = \begin{cases} 0, & v < 2, \\ 2v^{-2}, & 2 \leq v \leq 4, \\ \frac{1}{2}v^{-3/2}, & v > 4. \end{cases} \\ 1 - \frac{1}{\sqrt{v}}, & v > 4, \end{cases}$$

Alternatively, we can proceed as above but instead write F_Z in terms of F_X . Note that we showed

$$\text{above that } F_Z(v) = \begin{cases} 0, & v < 2 \\ F_X(v/2) - F_X(-v/2), & 2 \leq v \leq 4, \\ F_X(\sqrt{v}) - F_X(-\sqrt{v}), & v > 4. \end{cases} \text{ The chain rule for differentiation gives}$$

$$\text{us that } f_Z(v) = \begin{cases} 0, & v < 2, \\ \frac{1}{2} [f_X(\frac{v}{2}) + f_X(-\frac{v}{2})] = \frac{1}{2(v/2)^2} = 2v^{-2} & 2 \leq v \leq 4, \text{ just as before.} \\ \frac{1}{2\sqrt{v}} [f_X(\sqrt{v}) + f_X(-\sqrt{v})] = \frac{1}{2}v^{-3/2}, & v > 4, \end{cases}$$

4. [Random spheres]

(a) From the definition of expectation, $E[R] = \int_0^1 u \cdot 2u du = \frac{2}{3} u^3 \Big|_0^1 = \frac{2}{3}$.

(b) The volume of a sphere with radius R is given by $\text{Volume} = \frac{4}{3}\pi R^3$. Therefore, $E[\text{Volume}] = E[\frac{4}{3}\pi R^3] = \frac{4}{3}\pi E[R^3]$. LOTUS gives $E[R^3] = \int_0^1 u^3 \cdot 2u du = \frac{2}{5} u^5 \Big|_0^1 = \frac{2}{5}$. We see that $\frac{4}{3}\pi (E[R])^3 = \frac{4}{3}\pi (\frac{2}{3})^3$, the volume of a sphere of average radius, *does not* equal $\frac{4}{3}\pi \frac{2}{5} = E[\frac{4}{3}\pi R^3]$, the average volume of a sphere.

(c) The surface area of a sphere of radius R is given by $4\pi R^2$. Therefore, $E[\text{Surface Area}] = E[4\pi R^2] = 4\pi E[R^2]$. LOTUS gives $E[R^2] = \int_0^1 u^2 \cdot 2u du = \frac{1}{2} u^4 \Big|_0^1 = \frac{1}{2}$. We see that $4\pi (E[R])^2 = 4\pi (\frac{2}{3})^2$, the surface area of a sphere of average radius *does not* equal $4\pi \frac{1}{2} = 4\pi E[R^2]$, the average surface area of a sphere.

(d) Since $f(u) > 0$ only in the interval $(0, 1)$ we can use proposition 2.1 and write the expectation of any function of R as

$$E[g(R)] = \int_{-\infty}^{\infty} g(u)f(u)du = \int_0^1 g(u)f(u)du.$$

Now, for all $u \in (0, 1)$ we have $u > u^2 > u^3$. Therefore, for all $u \in (0, 1)$ we have $uf(u) \geq u^2 f(u) \geq u^3 f(u)$ where equality holds only when $f(u) = 0$. Thus, we have

$$\int_0^1 uf(u)du \geq \int_0^1 u^2 f(u)du \geq \int_0^1 u^3 f(u)du,$$

and the equality holds only when $f(u) = 0$ for all $u \in (0, 1)$. But, since $f(u)$ is a pdf, it cannot be zero everywhere. Thus,

$$\int_0^1 u f(u) du = E[R] > \int_0^1 u^2 f(u) du = E[R^2] > \int_0^1 u^3 f(u) du = E[R^3].$$

5. [Hypothesis testing]

- (a) We have $f_0(u) = \frac{1}{a\sqrt{2\pi}} e^{-\frac{u^2}{2a^2}}$ and $f_1(u) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{u^2}{2b^2}}$ and $\pi_0 = \pi_1 \Rightarrow \pi_0 = \frac{1}{2}$. Thus, using the theorem of total probability we have $f(u) = \frac{1}{2} \frac{1}{a\sqrt{2\pi}} e^{-\frac{u^2}{2a^2}} + \frac{1}{2} \frac{1}{b\sqrt{2\pi}} e^{-\frac{u^2}{2b^2}}$. This is *not* a Gaussian pdf.

$$(b) \Lambda(u) = \frac{\frac{1}{b\sqrt{2\pi}} e^{-\frac{u^2}{2b^2}}}{\frac{1}{a\sqrt{2\pi}} e^{-\frac{u^2}{2a^2}}} = \frac{a}{b} e^{-\frac{u^2}{2b^2} - \frac{u^2}{2a^2}} = \frac{a}{b} e^{-\frac{u^2}{2} \left(\frac{1}{b^2} + \frac{1}{a^2} \right)}$$

- (c) The maximum-likelihood decision rule is to choose H_1 when $\Lambda(u) > 1$. Thus, by taking the natural logarithm of both sides of this inequality we obtain the rule: If $\ln \frac{a}{b} - \frac{u^2}{2} \left(\frac{1}{b^2} + \frac{1}{a^2} \right) > 0$ choose H_1 . After a bit of algebra, we derive the rule that H_1 should be chosen when

$$\ln \frac{a}{b} > \frac{u^2}{2} \left(\frac{1}{b^2} + \frac{1}{a^2} \right) = \frac{u^2}{2} \frac{a^2 + b^2}{a^2 b^2} \Rightarrow 2 \ln \frac{a}{b} \left(\frac{a^2 b^2}{a^2 + b^2} \right) < u^2 \Rightarrow K = ab \sqrt{\frac{\ln a^2 - \ln b^2}{a^2 - b^2}} < |u|$$

Note that the inequality is reversed in the second step because $a^2 - b^2 < 0$. Also note that the value of K is fixed once a and b are specified. The false alarm probability is $P\{|X| > K \mid H_0\} = \int_{-K}^{-\infty} f_0(u) du + \int_K^{\infty} f_0(u) du = \Phi(-K/a) + 1 - \Phi(K/a) = 1 - \Phi(K/a) + 1 - \Phi(K/a) = 2(1 - \Phi(K/a)) = 2Q(K/a)$. The missed detection probability is given by $P\{|X| < K \mid H_1\} = \int_{-K}^K f_1(u) du = \Phi(K/b) - \Phi(-K/b) = \Phi(K/b) - (1 - \Phi(K/b)) = 1 - Q(K/b) - Q(K/b) = 1 - 2Q(K/b)$.

6. [Target recognition: Gaussian noise]

- (a) In the case where the variances are equal, the threshold is simply $(\mu_1 + \mu_0)/2 = 62.5$, and thus, the maximum likelihood decision is that a tank is present.

- (b) In general, the likelihood ratio is given by $\Lambda(u) = \sqrt{\frac{\sigma_0^2}{\sigma_1^2}} \exp\left(\frac{(u-25)^2}{2\sigma_0^2} - \frac{(u-100)^2}{2\sigma_1^2}\right)$. For

these variances we have $\Lambda(75) = \sqrt{\frac{20}{40}} \exp\left(\frac{(75-25)^2}{40} - \frac{(75-100)^2}{80}\right) = \sqrt{\frac{1}{2}} \exp(54.6875) > 1$.

Therefore we decide that the tank is present.

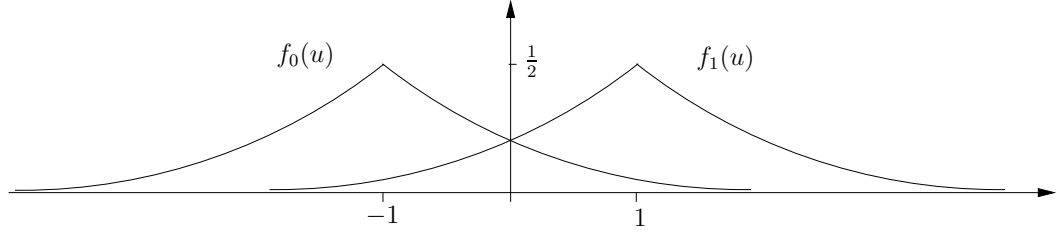
- (c) Here, $\Lambda(75) = \sqrt{\frac{50}{10}} \exp\left(\frac{(75-25)^2}{100} - \frac{(75-100)^2}{20}\right) = \sqrt{5} \exp(-6.25) < 1$. Therefore, we decide that no tank is present.

7. [Target detection using the Laplacian pdf]

- (a) Since $|u-1| = \begin{cases} -u+1, & u < 1, \\ u-1, & u \geq 1, \end{cases}$ and $|u+1| = \begin{cases} -u-1 & u < -1 \\ u+1 & u \geq -1, \end{cases}$, we have that $f_1(u) =$

$$\frac{1}{2} e^{-|u-1|} = \begin{cases} \frac{1}{2} e^{u-1}, & u < 1, \\ \frac{1}{2} e^{-u+1}, & u \geq 1, \end{cases} \text{ and } f_0(u) = \frac{1}{2} e^{-|u+1|} = \begin{cases} \frac{1}{2} e^{u+1}, & u < -1, \\ \frac{1}{2} e^{-u-1}, & u \geq -1. \end{cases}$$

The pdfs attain maximum value $\frac{1}{2}$ at ± 1 and are symmetric about their maximum values. From this we can sketch the pdfs as shown below.



- (b) From the sketch in part (a), it is easy to see that the maximum likelihood decision rule is to decide in favor of H_0 when $X < 0$ and in favor of H_1 when $X > 0$.
- (c) The false alarm probability is the probability that H_1 is decided when H_0 is true, i.e., when $X > 0$ but H_0 is true. Thus,

$$P_{FA} = \int_0^\infty f_0(u) du = \int_0^\infty \frac{1}{2} e^{-|u+1|} du = \int_0^\infty \frac{1}{2} e^{-u-1} du = \frac{1}{2} e^{-1} \int_0^\infty e^{-u} du = \frac{1}{2} e^{-1}.$$

By symmetry, we see that the probability of missed detection is also given by $P_{MD} = \frac{1}{2} e^{-1}$ (of course, one could always compute $\int_{-\infty}^0 \frac{1}{2} e^{-|u-1|} du = \int_{-\infty}^0 \frac{1}{2} e^{-(u-1)} du = \frac{1}{2} e^{-1} \int_{-\infty}^0 \frac{1}{2} e^u du = \frac{1}{2} e^{-1}$).

$$(d) \Lambda(u) = \frac{e^{-|u-1|}}{e^{-|u+1|}} = \begin{cases} \frac{e^{u-1}}{e^{u+1}} = e^{-2}, & u < -1, \\ \frac{e^{u-1}}{e^{-u-1}} = e^{2u}, & |u| \leq 1, \\ \frac{e^{-u+1}}{e^{-u-1}} = e^2, & 1 < u. \end{cases} \quad \text{Simply plugging in the values for } u \text{ we obtain}$$

$$\Lambda(-4) = \Lambda(-3) = \Lambda(-2) = \Lambda(-1) = e^{-2}; \Lambda(0) = 1; \text{ and } \Lambda(1) = \Lambda(2) = \Lambda(3) = \Lambda(4) = e^2.$$

- (e) Given the priors π_0 and π_1 , the Bayes decision rule can be expressed as if $\Lambda(u) > \frac{\pi_0}{\pi_1}$, then choose H_1 , and if $\Lambda(u) < \frac{\pi_0}{\pi_1}$, then choose H_0 . Clearly, if $\frac{\pi_0}{\pi_1} > e^2$, then $\Lambda(u)$ can never be greater than $\frac{\pi_0}{\pi_1}$, and the Bayes rule will always choose H_0 . Similarly, if $\frac{\pi_0}{\pi_1} < e^{-2}$, $\Lambda(u)$ can never be less than $\frac{\pi_0}{\pi_1}$, and the Bayes rule will always choose H_1 . When $e^{-2} < \frac{\pi_0}{\pi_1} < e^2$, we have $e^{2u} > \frac{\pi_0}{\pi_1} \rightarrow H_1$. By taking the natural log of both sides we have $e^{2u} > \frac{\pi_0}{\pi_1}$ when $2u > \ln \frac{\pi_0}{\pi_1}$ so we choose H_1 when $X > \ln \sqrt{\frac{\pi_0}{\pi_1}}$.

- (f) If $\pi_0 = 2\pi_1$, then we have $\pi_0 = 2/3$ and $\pi_1 = 1/3$ and $e^{-2} < \frac{\pi_0}{\pi_1} = 2 < e^2$. As above, the Bayes decision rule chooses H_1 when $u > \ln \sqrt{2}$. The false alarm probability is given by

$$P_{FA} = \int_{X > \ln \sqrt{2}} f_0(u) du = \frac{1}{2} \int_{\ln \sqrt{2}}^\infty e^{-u-1} du = \frac{1}{2} e^{-1} (e^{-\ln \sqrt{2}}) = \frac{e^{-1}}{2\sqrt{2}}.$$

$$\text{Similarly, } P_{MD} = \int_{X < \ln \sqrt{2}} f_1(u) du = \frac{1}{2} \int_{-\infty}^{\ln \sqrt{2}} \frac{1}{2} e^{u-1} du = \frac{1}{2} e^{-1} e^{\ln \sqrt{2}} = \sqrt{2} \frac{1}{2} e^{-1} = \frac{e^{-1}}{\sqrt{2}}.$$

$$\text{Thus, the average error probability is } \pi_0 P_{FA} + \pi_1 P_{MD} = \frac{2}{3} \frac{e^{-1}}{2\sqrt{2}} + \frac{1}{3} \frac{e^{-1}}{\sqrt{2}} = \frac{\sqrt{2} e^{-1}}{3}.$$

- (g) This decision rule always chooses H_0 , and thus the probability of error is just the probability that H_1 is the true hypothesis: π_1 .

- (h) $\pi_0 > \frac{e^2}{e^2 + 1} \Rightarrow \pi_1 = 1 - \pi_0 < 1 - \frac{e^2}{e^2 + 1} = \frac{1}{e^2 + 1} \Rightarrow \frac{\pi_0}{\pi_1} > \frac{e^2(e^2 + 1)}{e^2 + 1} = e^2$. Thus, as discussed above in part (e), the Bayes decision rule always chooses H_0 .