

ECE 313: Problem Set 9: Solutions

Exponential distribution, Poisson random processes, Gaussian random variables

1. [Lifetime of a network with two parallel links]

- (a) Since T_1 is exponentially distributed with parameter λ , $P\{T_1 > t\} = \int_t^\infty \lambda \exp(-\lambda t) dt = \exp(-\lambda t)$. Furthermore, for $s, t > 0$, $P(T_1 > s + t | T_1 > s) = \frac{P\{T_1 > s+t\}}{P\{T_1 > s\}} = \frac{\exp(-\lambda(s+t))}{\exp(-\lambda s)} = \exp(-\lambda t)$. Thus, $P(T_1 > s + t | T_1 > s) = P\{T_1 > t\}$, a property known as the memoryless property of the exponential distribution. For this problem, $\lambda = 1$, so $P\{T_1 > 1\} = P(T_1 > 2 | T_1 > 1) = e^{-1} \approx 0.36788$.
- (b) $\{T > t\} = \{T_1 > t\} \cup \{T_2 > t\}$ and the events $\{T_1 > t\}$ and $\{T_2 > t\}$ are independent (because T_1 and T_2 are independent) so
 $P\{T > t\} = P\{T_1 > t\} + P\{T_2 > t\} - P\{T_1 > t\}P\{T_2 > t\} = 2e^{-t} - e^{-2t}$.
- (c) Using the answer to part (b), we have $P\{T > 1\} = 2e^{-1} - e^{-2} \approx 0.6004$, and $P(T > 2 | T > 1) = \frac{P\{T > 2\}}{P\{T > 1\}} = \frac{2e^{-2} - e^{-4}}{2e^{-1} - e^{-2}} \approx 0.4203$. We see $P\{T > 1\} > P(T > 2 | T > 1)$. In particular, the distribution of T is not memoryless. Intuitively, the reason is that even if the network has not failed by time $t = 1$, one of the two links might have failed.
- (d) $f_T(t) = -(P\{T > t\})' = 2(e^{-t} - e^{-2t})$ for $t \geq 0$. Also, $f_T(t) = 0$ for $t < 0$.
- (e) $E[T] = \int_0^\infty P\{T > t\} dt = 2 - 0.5 = 1.5$. Or $E[T] = \int_0^\infty t f_T(t) dt = 2 - 0.5 = 1.5$.

2. [Lifetime of a network with four links]

- (a) $\{T > t\} = (\{T_1 > t\} \cup \{T_2 > t\})(\{T_3 > t\} \cup \{T_4 > t\})$ so $P\{T > t\} = (2e^{-t} - e^{-2t})^2 = 4e^{-2t} - 4e^{-3t} + e^{-4t}$. (Another approach is to start with $\{T \leq t\} = (\{T_1 \leq t\}\{T_2 \leq t\}) \cup (\{T_3 \leq t\}\{T_4 \leq t\})$, to get $F_T(t) = 2(1 - e^{-t})^2 - (1 - e^{-t})^4$.)
- (b) $f_T(t) = -(P\{T > t\})' = 8e^{-2t} - 12e^{-3t} + 4e^{-4t}$ for $t \geq 0$. Another expression, showing that f_T is nonnegative, is $f_T(t) = 4e^{-2t}(2 - e^{-t})(1 - e^{-t})$ for $t \geq 0$. Of course, $f_T(t) = 0$ for $t < 0$.
- (c) $E[T] = \int_0^\infty P\{T > t\} dt = 2 - \frac{4}{3} + \frac{1}{4} = \frac{11}{12}$. Or $E[T] = \int_0^\infty t f_T(t) dt = 2 - \frac{4}{3} + \frac{1}{4} = \frac{11}{12}$.

3. [Arrival times for a Poisson process]

- (a) $P(N(t) = i) = \frac{e^{-2t}(2t)^i}{i!}$ for nonnegative integers i (Poisson distribution, mean $2t$)
- (b) $P(E_1) = P(N(3.5) = 0) = \frac{e^{-7}(7)^0}{0!} = e^{-7} = 0.00091$
 $P(E_2) = 0.00091$ since E_2 is the same event as E_1
 $P(E_3) = P(N(3.5) \leq 2) = P(N(3.5) = 0) + P(N(3.5) = 1) + P(N(3.5) = 2) = e^{-7}(1 + 7 + \frac{7^2}{2}) = 0.0296$
 $P(E_4) = 0.0296$ since E_4 is the same event as E_3
 $P(E_5) = P(N(t) \leq 2) = P(N(t) = 0) + P(N(t) = 1) + P(N(t) = 2) = e^{-2t}(1 + 2t + \frac{(2t)^2}{2})$
 $P(E_6) = 1 - P(E_5) = 1 - e^{-2t}(1 + 2t + \frac{(2t)^2}{2})$
- (c) Differentiating yields $f(t) = e^{-2t}(2(1 + 2t + \frac{(2t)^2}{2}) - 2 - 4t) = e^{-2t}(2t)^2 = e^{-2t} \frac{2^3 t^2}{2}$, which indeed is the gamma density with parameters ($i = 3, \lambda = 2$).

- (d) The expected time between arrivals is $1/\lambda$, so the expected time until the tenth arrival is $10/\lambda = 5$ minutes.

4. [Poisson process probabilities]

- (a) The numbers of arrivals in the disjoint intervals are independent, Poisson random variables with mean λ . Thus, the probability is $(\lambda e^{-\lambda})^3 = \lambda^3 e^{-3\lambda}$.
- (b) We wish to determine the probability of the event $A = \{N(0, 2] = 2, N(1, 3] = 2\}$. A is the union of the disjoint events B_{020} , B_{111} , and B_{202} , where $B_{ijk} = \{N(0, 1] = i, N(1, 2] = j, N(2, 3] = k\}$. So $P(A) = P(B_{020}) + P(B_{111}) + P(B_{202}) = e^{-\lambda}(\frac{\lambda^2}{2}e^{-\lambda})e^{-\lambda} + (\lambda e^{-\lambda})^3 + (\frac{\lambda^2}{2}e^{-\lambda})e^{-\lambda}(\frac{\lambda^2}{2}e^{-\lambda}) = (\frac{\lambda^2}{2} + \lambda^3 + \frac{\lambda^4}{4})e^{-3\lambda}$.
- (c) $P(B_{020}|A) = \frac{P(B_{020}A)}{P(A)} = \frac{P(B_{020})}{P(A)} = \frac{\lambda^2/2}{(\lambda^2/2 + \lambda^3 + \lambda^4/4)} = 2/(2 + 4\lambda + \lambda^2)$. (We have applied Bayes rule, which in this case is the definition of the conditional probability $P(B_{020}|A)$ and use of the law of total probability for finding $P(A)$.)

5. [Calculating probabilities of events involving a Gaussian random variable]

- (a) In parts (a)-(c) we use the fact that $\frac{X-10}{4}$ is a $N(0,1)$ random variable.
 $P(X \geq 15) = P(\frac{X-10}{4} \geq \frac{15-10}{4}) = Q(\frac{15-10}{4}) = Q(1.25) = 1 - \Phi(1.25) = 0.1056$.
- (b) $P(X \leq 5) = P(\frac{X-10}{4} \leq \frac{5-10}{4}) = P(\frac{X-10}{4} \leq -1.25) = Q(1.25) = 0.1056$.
- (c) $P(X^2 \geq 400) = P(X \geq 20) + P(X \leq -20) = P(\frac{X-10}{4} \geq 2.5) + P(\frac{X-10}{4} \leq -7.5) = Q(2.5) + Q(7.5) \approx Q(2.5) = 1 - \Phi(2.5) = 0.0062$ (Note: $Q(7.2) < 10^{-12}$.)
- (d) $P(X = 2) = 0$ because X is a continuous type random variable.

6. [Normal approximation to central term of binomial distribution]

- (a) Note that $\text{Var}(X) = np(1-p) = n/4$.

$$\begin{aligned} P\left\{X = \frac{n}{2}\right\} &= P\left\{\frac{n-1}{2} \leq X \leq \frac{n+1}{2}\right\} \\ &= P\left\{-\frac{1}{2} \leq X - \frac{n}{2} \leq \frac{1}{2}\right\} \\ &= P\left\{-\frac{1}{\sqrt{n}} \leq \frac{X - n/2}{\sqrt{n/4}} \leq \frac{1}{\sqrt{n}}\right\} \\ &\approx \Phi\left(\frac{1}{\sqrt{n}}\right) - \Phi\left(-\frac{1}{\sqrt{n}}\right) = 2\left(\Phi\left(\frac{1}{\sqrt{n}}\right) - 0.5\right) \end{aligned}$$

- (b) Rather than using the hint, we use the fact that $\Phi(0) = 0.5$ and the definition of derivative. Also, the derivative of Φ is the standard normal density, so that $\Phi'(0) = \frac{1}{\sqrt{2\pi}}$. Thus,

$$\sqrt{n} \times 2\left(\Phi\left(\frac{1}{\sqrt{n}}\right) - 0.5\right) = 2\left(\frac{\Phi\left(\frac{1}{\sqrt{n}}\right) - \Phi(0)}{\frac{1}{\sqrt{n}}}\right) \xrightarrow{n \rightarrow \infty} 2\Phi'(0) = \sqrt{\frac{2}{\pi}} \approx 0.7979$$

- (c) For $n = 30$, we have

true value	$P\{X = \frac{n}{2}\} = \binom{n}{n/2} 2^{-n}$	0.144464
approximation from (a)	$\Phi(0.1825741) - \Phi(-0.1825741)$	0.144868
approximation from (b)	$\sqrt{\frac{2}{n\pi}}$	0.145673