

## ECE 313: Problem Set 8: Solutions

### Cumulative Distribution Functions, Continuous Random Variables

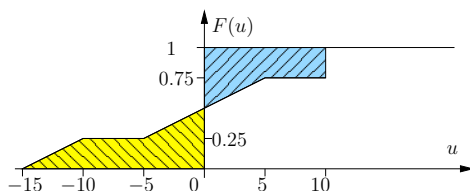
#### 1. [Maximization of the newsboy's profit: Generalization of Problem 4 of Problem Set 4]

- (a) The newsboy sells all his papers if and only if the demand is  $H$  or more, i.e., if and only if the event  $\{\mathbb{X} \geq H\}$  occurs.  $P\{\mathbb{X} \geq H\} = 1 - P\{\mathbb{X} \leq H - 1\} = 1 - F(H - 1)$ .
- (b)  $P\{\text{unable to sell extra paper}\} = P\{\text{demand is at most } H\} = P\{\mathbb{X} \leq H\} = F(H)$ .
- (c) The newsboy is unable to sell the extra paper with probability  $F(H)$  and thus loses  $c_1 - c_3$  cents with this probability, or is able to sell the extra paper with probability  $1 - F(H)$  and thus makes  $c_2 - c_1$  cents with this probability. Hence, his average additional profit from the  $(H + 1)$ -th paper is  $A(H + 1) = (c_2 - c_1)(1 - F(H)) - (c_1 - c_3)F(H) = (c_2 - c_1) - F(H)(c_2 - c_3)$ .
- (d) Since  $c_2 - c_3 > c_2 - c_1 > 0$  and since  $F(H)$  is a non-decreasing function of  $H$ , it follows that  $A(H + 1)$  is a non-increasing function of  $H$ . Furthermore,  $A(H + 1) < 0$  for all  $H$  such that  $F(H) > \frac{c_2 - c_1}{c_2 - c_3}$ . Note that  $\frac{c_2 - c_1}{c_2 - c_3} < 1$  while  $F(\infty) = 1$  and so  $F(H) > \frac{c_2 - c_1}{c_2 - c_3}$  for sufficiently large values of  $H$ .
- (e) Buying one extra paper over and above his daily  $H$  *increases* the newsboy's average daily profit if  $A(H + 1) > 0$ , makes no difference if  $A(H + 1) = 0$ , and *reduces* his average daily profit if  $A(H + 1) < 0$ . So, he should buy  $H_0$  papers where  $H_0$  is the smallest integer such that  $A(H_0 + 1) \leq 0$ , that is,  $F(H_0) \geq \frac{c_2 - c_1}{c_2 - c_3}$ . If  $F(H_0) = \frac{c_2 - c_1}{c_2 - c_3}$  exactly, then  $A(H_0 + 1) = 0$ , and so buying the extra paper gives the newsboy zero average additional profit. So the newsboy avoids working harder than he has to. Wouldn't you?
- (f) A spreadsheet shows that  $F(H_0) \geq \frac{60 - 35}{60 - 25} = 0.714\dots$  for  $H_0 \geq 63$ . So, the newsboy should buy 63 papers to make the maximum average daily profit of 1442.063\dots cents. Buying exactly 60 papers, the average number that he sells per day, yields a slightly smaller average profit of 1431.77\dots cents. On the other hand, buying 100 papers yields him an average daily profit of \$11 exactly. This is an easier calculation: he makes a total of  $25\mathbb{X}$  cents on his sales and loses  $10(100 - \mathbb{X})$  on the papers he recycles for a net gain of  $35\mathbb{X} - 1000$  cents, and  $E[35\mathbb{X} - 1000] = 35E[\mathbb{X}] - 1000 = 35 \times 60 - 1000 = 2100 - 1000 = 1100$ .

#### 2. [Expected values from CDFs]

$$E[\mathbb{X}] = \int_0^\infty [1 - F(u)] du - \int_{-\infty}^0 F(u) du = \text{area shaded } \textcolor{blue}{\square} - \text{area shaded } \textcolor{yellow}{\square} \text{ in figure}$$

$$= \left(10 \times \frac{1}{4} + \frac{1}{2} \times 5 \times \frac{1}{4}\right) - \left(10 \times \frac{1}{4} + 2 \times \frac{1}{2} \times 5 \times \frac{1}{4}\right) = -\frac{5}{8}.$$



#### 3. [Exercises on properties of pdfs]

If  $f(u)$  is a nonnegative (or nonpositive) function with finite nonzero area  $A$ , then  $A^{-1} \cdot f(u)$  is a valid pdf. This does not work if  $f(u)$  takes on both positive and negative values.

- (a)  $f(u) = 2u$ ,  $0 < u < 1$  is a valid pdf.
- (b)  $f(u) = |u|$ ,  $|u| < \frac{1}{2}$  is not a valid pdf, but  $4 \cdot f(u)$  is.
- (c)  $f(u) = 1 - |u|$ ,  $|u| < 1$  is a valid pdf.
- (d)  $f(u) = \ln u$ ,  $0 < u < 1$  is not a valid pdf but  $-f(u)$  is.
- (e)  $f(u) = \ln u$ ,  $0 < u < 2$  is not a valid pdf, nor is  $C \cdot f(u)$  a valid pdf for any choice of  $C$ .
- (f)  $f(u) = \frac{2}{3}(u - 1)$ ,  $0 < u < 3$ , is not a valid pdf, nor is  $C \cdot f(u)$  a valid pdf for any choice of  $C$ .
- (g)  $f(u) = \exp(-2u)$ ,  $u > 0$  is not a valid pdf but  $2 \cdot f(u)$  is.
- (h)  $f(u) = 4e^{-2u} - e^{-u}$ ,  $u > 0$ , is not a valid pdf, nor is  $C \cdot f(u)$  a valid pdf for any choice of  $C$ .
- (i)  $f(u) = \exp(-|u|)$ ,  $|u| < 1$  is not a valid pdf but  $e/[2(e - 1)] \cdot f(u)$  is.

4. [Lifetime of a device].

- (a)  $P\{\mathbb{X} > 20\} = \int_{20}^{\infty} \frac{10}{u^2} du = \left. \frac{-10}{u} \right|_{20}^{\infty} = \frac{1}{2}$ .
- (b)  $F_{\mathbb{X}}(u_0) = \int_{-\infty}^{u_0} f_{\mathbb{X}}(u) du = \int_{10}^{u_0} \frac{10}{u^2} du = \left. \frac{-10}{u} \right|_{10}^{u_0} = 1 - \frac{10}{u_0}$ . Note that  $F_{\mathbb{X}}(20) = \frac{1}{2}$ ; it should be!
- (c)  $P\{\mathbb{X} > 15\} = 1 - F_{\mathbb{X}}(15) = \frac{10}{15} = \frac{2}{3}$ . Assuming that the device failures are independent, the probability that at least 3 of the 6 last for 15 hours (i.e., at most 3 fail) is a binomial probability:

$$P\{\text{at least 3 of 6 devices work for 15 hours}\} = P\{\text{at most 3 failures}\} = \sum_{i=0}^3 \binom{6}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{6-i}.$$

5. [Maximizing profits, Part II]

- (a)  $P\{\text{demand can be satisfied}\} = P\left\{\mathbb{X} \leq \frac{1}{2}\right\} = \int_0^{0.5} 5(1-u)^4 du = \left. -(1-u)^5 \right|_0^{0.5} = \frac{31}{32}$ .
- (b) We want the minimum value of  $C$  such that  $P\{\mathbb{X} > C\} \leq 10^{-5}$ . But,
 
$$P\{\mathbb{X} > C\} = \int_C^1 5(1-u)^4 du = \left. -(1-u)^5 \right|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9.$$
- (c)  $\mathbb{Y} = \min\{\mathbb{X}, C\} = \begin{cases} \mathbb{X}, & \text{if } \mathbb{X} \leq C, \\ C, & \text{if } \mathbb{X} > C. \end{cases}$

The weekly *gross* profit is  $\$640\mathbb{Y}$  and the *average* weekly gross profit is

$$E[640\mathbb{Y}] = \int_0^C 5(1-u)^4(640u) du + \int_C^1 5(1-u)^4(640C) du = \frac{640}{6} [1 - (1-C)^6].$$

As a function of  $C$ , this increases from 0 at  $C = 0$  to  $640/6$  when  $C = 1$ .

- (d) The average *net* profit is  $\$E[(640\mathbb{Y} - 20C)] = E[(640\mathbb{Y}) - 20C] = \frac{640}{6} [1 - (1-C)^6] - 20C$  whose maximum can be found by the usual calculus method to be at  $C = \frac{1}{2}$ . Thus, a 500 gallon tank that costs \$10 per week to rent gives an average gross profit of  $\$630/6$  and the maximum average net profit of  $\$570/6$  per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of  $\$640/6$  per week, but a smaller net profit of  $\$520/6$  per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only  $\frac{1}{32}$  when  $C = \frac{1}{2}$ .