

ECE 313: Problem Set 7: Solutions

Independence, Reliability, and CDFs

1. [Network capacity and failure]

- (a) The probability of success is the probability that one of the first two parallel links will function AND the middle link will function AND one of the last two parallel links will function: $P(\text{success}) = (p + p - p^2)(p)(p + p - p^2) = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{9}{32}$.
- (b) If a message can be sent, then 10 messages can be sent (any path through the network has capacity at least 10). More than 10 messages can never be sent, since the capacity of the middle link is only 10. Thus, X has two possible values, and using the results of part (a) we have $p_X(10) = P(\text{success}) = \frac{9}{32}$, $p_X(0) = P(\text{failure}) = 1 - \frac{9}{32} = \frac{23}{32}$.
- (c) Simply use the definition of expectation $E[X] = (0)(p_X(0)) + (10)(p_X(10)) = \frac{90}{32} = \frac{45}{16}$.

2. [Practical lessons for gamblers]

- (a) $P(\text{wins on first roll}) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$. $P(\text{loses on first roll}) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{9}$.
 $P(\text{establishes point } i \text{ on first roll}) = \frac{6 - |7 - i|}{36}, i = 4, 5, 6, 8, 9, 10$. Check: probabilities sum to 1.
- (b) If the shooter's point is i , then $P(\text{makes point } i \mid \text{point is } i) = \frac{6 - |7 - i|}{36}$ while $P(\text{craps out}) = \frac{6}{36}$ so that $P(\text{game ends}) = \frac{12 - |7 - i|}{36}$. Else, the shooter rolls the dice again. Now, given disjoint events A and B , the probability that A occurs before B is the probability that a compound event of the form $(C, C, \dots, C, A)$ occurs where $C = (A \cup B)^c$. Thus,

$$P(A \text{ occurs before } B) = P(A) + P(C)P(A) + [P(C)]^2P(A) + \dots = \frac{P(A)}{1 - P(C)} = \frac{P(A)}{P(A) + P(B)}.$$

Hence, $P(\text{shooter wins} \mid \text{point is } i) = \frac{6 - |7 - i|}{12 - |7 - i|}$ which equals $\frac{3}{9}, \frac{4}{10}, \frac{5}{11}, \frac{5}{11}, \frac{4}{10}, \frac{3}{9}$ according as i equals 4, 5, 6, 8, 9, 10. The theorem of total probability now gives that

$$P(\text{shooter wins}) = \frac{2}{9} + 2 \left[\frac{3}{9} \times \frac{3}{36} + \frac{4}{10} \times \frac{4}{36} + \frac{5}{11} \times \frac{5}{36} \right] = \frac{244}{495} = 0.492929 \dots$$

- (c) Conditioned on the shooter's point being i , the number of *additional* dice rolls till the game ends is a geometric random variable with parameter $p = P(\text{game ends}) = \frac{12 - |7 - i|}{36}$ and conditional mean $\frac{36}{12 - |7 - i|}$. The expected number of dice rolls in a game of craps is thus

$$E[X] = \frac{3}{9} \times 1 + 2 \left[\left(1 + \frac{36}{9}\right) \frac{3}{36} + \left(1 + \frac{36}{10}\right) \frac{4}{36} + \left(1 + \frac{36}{11}\right) \frac{5}{36} \right] = 3 \frac{62}{165}.$$

- (d) If the shooter's point is 8, then his chances of making the point the hard way are $\frac{1}{36}$ while the probability of making the point the easier way (or crapping out) are $\frac{6+4}{36}$. Thus, the shooter's probability of winning the hard way is $\frac{1}{1+10} = \frac{1}{11}$. Thus, the odds are perfectly fair. Ten times out of eleven, you will lose the dollar that you wager while one time out of eleven, you will win ten dollars.

3. [Cumulative distribution functions: Part I]

- (a) $P[X \leq 1] = F(1) = 0.55$
- (b) $P[X > 1] = 1 - P[X \leq 1] = 0.45$
- (c) $P[|X| \leq 3] = F(3) - F(-3) = 0.65 - 0.35 = 0.3$
- (d) $P[X^2 \leq 9] = P[|X| \leq 3] = 0.3$

4. [Voting Blocks]

- (a) The bill passes if the Conservative Republicans and at least one other group votes for it, i.e. the event $A \cap (B \cup C \cup D)$ occurs, or if all groups except the Conservative Republicans vote for it, i.e. the event $A^c \cap B \cap C \cap D$ occurs. Note that these are disjoint events (Why?). Now, $(B \cup C \cup D)^c = B^c \cap C^c \cap D^c$ by DeMorgan's theorem, and thus we can partition A into two disjoint events as follows: $A = A \cap (B \cup C \cup D) \cup (A \cap (B^c \cap C^c \cap D^c))$. Hence, we see that

$$P(A \cap (B \cup C \cup D)) = P(A) - P(A \cap B^c \cap C^c \cap D^c) = P(A) - P(A)P(B^c)P(C^c)P(D^c)$$

by independence. Since $P(A^c \cap B \cap C \cap D) = P(A^c)P(B)P(C)P(D)$ by independence, we get

$$\begin{aligned} P(\text{bill passes}) &= P(A \cap (B \cup C \cup D)) + P(A^c \cap B \cap C \cap D) \\ &= P(A) - P(A)P(B^c)P(C^c)P(D^c) + P(A^c)P(B)P(C)P(D) \\ &= 0.9 \times [1 - 0.4 \times 0.5 \times 0.8] + 0.1 \times 0.6 \times 0.5 \times 0.2 \\ &= 0.756 + 0.006 = 0.762. \end{aligned}$$

- (b) The Conservative Republicans must be in favor for the motion to override to pass. Furthermore, since politics makes strange bedfellows, either the Liberal Democrats must also be in favor or both the other groups must also be in favor. Hence,

$$\begin{aligned} P(\text{veto overridden}) &= P(E \cap (H \cup (F \cap G))) \\ &= P(E) \times P(H \cup (F \cap G)) \quad \text{by independence of } E \text{ and } H \cup (F \cap G) \\ &= P(E) \times [P(H) + P(F \cap G) - P(F \cap G \cap H)] \\ &= P(E) \times [P(H) + P(F)P(G) - P(F)P(G)P(H)] \quad \text{independence again!} \\ &= 0.99 \times [0.1 + 0.4 \times 0.6 - 0.6 \times 0.4 \times 0.1] = 0.31284. \end{aligned}$$

5. [Fault tolerance]

- (a) For the single gigantic car, each part fails with probability p . All the N parts of a given type must fail in order for the car to fail, and this occurs with probability p^N , and so the probability that at least one part of a given type is working is $1 - p^N$. The probability that at least one part of each of the M types is working is $(1 - p^N)^M$, and hence the probability of system failure is $1 - (1 - p^N)^M$.

On the other hand, if we have N cars each with M parts, then the probability that at least one part fails in a car is $1 - (1 - p)^M$. This is also the probability that the car itself has failed. But, we have N cars, and thus the probability that at least one car is in working condition (i.e. that not all have failed) is $(1 - (1 - p)^M)^N$.

- (b) For the single gigantic car, we want $1 - ((1 - 0.2^N)^5) < 10^{-3}$ which is achieved by choosing $N \geq 6$. For the N small cars, we want $(1 - (1 - 0.2)^5)^N < 10^{-3}$ which is achieved by choosing $N \geq 18$. Thus, a gigantic car with 6 engines, 6 transmissions, 6 brakes etc provides more reliable transmission than having 17 different compact cars.
- (c) If $M = 1000$, we get that $N \geq 9$ for the gigantic car while typical calculators blow up on the calculation for the number of small cars needed. The Unix high-precision utility *bc* gives that $1 - (1 - 0.2)^{1000} = 0.99 \dots 9986160344 \dots$ where there are eightyseven 9's preceding the 8616, and the number of cars needed is astronomical. Parking, of course, is impossible, but keep in mind that a gigantic car with 9 engines etc will also be very difficult to park!

Moral: It is better to replicate parts than to replicate systems.

6. [Failure rate function for discrete random variables]

- (a) Since $\{T = k\} \subset \{T \geq k\}$, the definitions of conditional probability and pmf yield $\lambda_T(k) = P(T = k | T \geq k) = \frac{P(T=k)}{P(T \geq k)} = \frac{p_T(k)}{\sum_{l=k}^{\infty} p_T(l)}$. Or, equivalently, $\lambda_T(k) = \frac{p_T(k)}{1 - \sum_{l=1}^{k-1} p_T(l)}$. (In particular, $\lambda_T(1) = p_T(1)$. Note that $\lambda_T(k)$ is only well-defined $P(T \geq k) > 0$.)
- (b) By definition, $\lambda_T(k_f) = 1$ if and only if $P(T = k_f) = P(T \geq k_f) > 0$. So $\lambda_T(k_f) = 1$ if and only if $P(T > k_f) = P(T \geq k_f) - P(T = k_f) = 0$ and $P(T = k_f) > 0$. Thus, the condition $\lambda_T(k_f) = 1$ means that T has a largest possible value, and that value is k_f .
- (c) For $1 \leq k \leq n$, $\lambda_T(k) = \frac{p_T(k)}{1 - \sum_{l=1}^{k-1} p_T(l)} = \frac{\frac{1}{n}}{1 - \frac{k-1}{n}} = \frac{1}{n-k+1}$. (That is, the first n values of λ_T are $\frac{1}{n}, \frac{1}{n-1}, \dots, \frac{1}{1}$.)
- (d) One solution is to first derive a general formula. Note that

$$\begin{aligned} P(T \geq k+1) &= P(\{T \geq k+1\} \cap \{T \geq k\}) \\ &= P(T \geq k+1 | T \geq k) P(T \geq k) \\ &= (1 - P(T = k | T \geq k)) P(T \geq k) \\ &= (1 - \lambda_T(k)) (P(T \geq k)). \end{aligned}$$

Starting with the initial condition $P(T \geq 1) = 1$, this can be used to solve for $P(T \geq k)$ for all k . The result is $P(T \geq k) = (1 - \lambda_T(1))(1 - \lambda_T(2)) \dots (1 - \lambda_T(k-1))$ for $k \geq 1$. Therefore,

$$\begin{aligned} P(T = k) &= P(\{T \geq k\} \cap \{T = k\}) \\ &= P(T \geq k) P(T = k | T \geq k) = P(T \geq k) \lambda_T(k) \\ &= (1 - \lambda_T(1))(1 - \lambda_T(2)) \dots (1 - \lambda_T(k-1)) \lambda_T(k). \end{aligned}$$

In particular, if $\lambda_T(k) = p$ for all $k \geq 1$, then $p_T(k) = (1 - p)^{k-1} p$ for $k \geq 1$. In other words, T has the geometric distribution with parameter p .

- (e) Fix $k \geq 0$. Consider conditional probabilities given the event $\{T \geq k\}$. Equivalently, consider conditional probabilities given that both links are still working at time k . Then the conditional probability that the network fails at time k , is the conditional probability that the first link fails at time k , plus the conditional probability that the second link fails at time k , minus the conditional probability that both links fail at time k . That is $\lambda_T(k) = \lambda_1(k) + \lambda_2(k) - \lambda_1(k)\lambda_2(k)$. (Note: If $\lambda_1(k)$ and $\lambda_2(k)$ are both very small, then $\lambda_T(k) \approx \lambda_1(k) + \lambda_2(k)$.)

7. [Cumulative distribution functions, Part II]

X takes on values 1, 2, 3, 4, 5, 6 with probabilities $\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}, \frac{6}{21}$ respectively. Hence,

