

ECE313: Problem Set 5: Solutions

Conditional Probability, Law of Total Probability, and Bayes' Formula

1. [Picking a random subset]

- (a) There are $\binom{n}{k}$ different subsets of size k , and since each is equally likely to be drawn, the probability of drawing a specific subset is just $\left[\binom{n}{k}\right]^{-1} = \frac{k \times (k-1) \times \cdots \times 2 \times 1}{n \times (n-1) \times \cdots \times (n-k+1)}$.
- (b) There are k "desirable" objects among the set of n objects and if we draw one at random, the probability that it is a "desirable" object is k/n , that is, $P(A_1) = k/n$. If the event A_1 has occurred, then there are only $k-1$ "desirable" objects left among the $n-1$. Thus, $P(A_2 | A_1) = (k-1)/(n-1)$, and so on. More generally, $P(A_i | A_1 A_2 \cdots A_{i-1}) = (k - (i-1))/(n - (i-1))$ since $i-1$ "desirable objects" have been drawn already. Finally,

$$P(A_1 A_2 \cdots A_k) = P(A_1)P(A_2 | A_1) \cdots P(A_k | A_1 A_2 \cdots A_{k-1}) = \frac{k \times (k-1) \times \cdots \times 2 \times 1}{n \times (n-1) \times \cdots \times (n-k+1)}.$$

2. [The beginning of spring training]

Since the pitcher can only pitch fast balls, curve balls, and sliders, we have that $P(F) + P(C) + P(S) = 1$, and since $P(C) = 2P(F)$, we conclude that $P(S) = 1 - 3P(F)$. The law of total probability gives

$$\begin{aligned} P(H) &= \frac{1}{4} = P(H | F)P(F) + P(H | C)P(C) + P(H | S)P(S) \\ &= P(H | F)P(F) + P(H | C) \cdot 2P(F) + P(H | S)(1 - 3P(F)) \\ &= \frac{2}{5}P(F) + \frac{1}{4} \cdot 2P(F) + \frac{1}{6}(1 - 3P(F)) \end{aligned}$$

from which we get that $P(F) = \frac{5}{24}$, $P(C) = \frac{10}{24}$, and $P(S) = \frac{9}{24}$.

3. [Exercises in Conditional Probability]

- (a) $P(A | B) = \frac{P(AB)}{P(B)} = \frac{P(ABC) + P(ABC^c)}{P(B)} = \frac{P(ABC)P(BC)}{P(BC)P(B)} + \frac{P(ABC^c)P(BC^c)}{P(BC^c)P(B)}$
 $= P(A | BC)P(C | B) + P(A | BC^c)P(C^c | B).$
- (b) $P(A | B^c) = 1 - P(A^c | B^c) = 0.6$.
 $P(A) = P(A | B)P(B) + P(A | B^c)P(B^c) = 0.3 \times 0.7 + 0.6 \times 0.3 = 0.39$.
 $P(B | A) = P(A | B)P(B)/P(A) = 0.3 \times 0.7/0.39 = 7/13$.

(c) $P(E \cap F) = P(F | E)P(E) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ and $P(E | F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{P(F)} = \frac{1}{3}$
 $\Rightarrow P(F) = \frac{3}{8}.$

(d) $P(G \cup H) = P(G) + P(H) - P(G \cap H) = \frac{4}{3} - P(G \cap H) \leq 1$ from which we deduce that
 $P(G \cap H) \geq \frac{1}{3}$ and hence $P(G | H) = \frac{P(G \cap H)}{P(H)} \geq \frac{1/3}{2/3} = \frac{1}{2}.$

4. [Polya's urn model: a classic result in probability theory]

- (a) This is sampling with replacement and we have that $P(R_1) = P(R_2) = \frac{r}{r+g}.$

- (b) $P(R_2) = P(R_2 | R_1)P(R_1) + P(R_2 | R_1^c)P(R_1^c) = \frac{r+c}{r+c+g} \times \frac{r}{r+g} + \frac{r}{r+c+g} \times \frac{g}{r+g} = \frac{r}{r+g}$
just as before, *and independent of the value of c !* Note that if the trials are repeated and $c > 0$, then the urn will contain $r + g + cn$ balls after n trials and yet the probability of drawing a red ball on the $(n + 1)$ -th trial will still be the same $\frac{r}{r+g}$ as before

(c) $P\{r + c \text{ red balls} | R_2\} = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{P(R_2 | R_1)P(R_1)}{P(R_2)} = P(R_2 | R_1) = \frac{r+c}{r+c+g}.$

5. **[Fred and Wilma learn the Law of Total Probability and escape from Guantanamo Bay]**

- (a) Fred wins the game if he tosses a Head on his first attempt (without Wilma even getting to touch the coin) and so obviously $P(F | H) = 1$ and $P(W | H) = 0$.
- (b) If Fred tosses a Tail on his first attempt, we have in essence a brand new game in which Wilma gets to toss first and wins if she tosses a Head. So her win probability at this point is exactly the same as that of the first tosser in the game, namely Fred. In other words, $P(W | H^c) = P(W | T) = P(F)$. Similarly, having tossed a Tail on his first try, Fred must now wait for Wilma to toss before he can toss again. If Wilma tosses a Head, she wins and he won't even get to toss the coin again. That is, having tossed a Tail, Fred now has the role of the second tosser in a brand-new game and his win probability is thus $P(F | H^c) = P(F | T) = P(W)$.
- (c) We have that

$$\begin{aligned} P(F) &= P(F | H)P(H) + P(F | H^c)P(H^c) = 1 \cdot p + P(W) \cdot q = p + qP(W), \\ P(W) &= P(W | H)P(H) + P(W | H^c)P(H^c) = 0 \cdot p + P(F) \cdot q = qP(F). \end{aligned}$$

It follows that $P(F) = p + q^2P(F)$ and hence $P(F) = \frac{p}{1-q^2} = \frac{1}{1+q}$ since $p = 1 - q$. Also, $P(W) = qP(F) = \frac{q}{1+q}$ just as before. Look Ma! No summing of series needed!

- (d) Yes, Fred and Wilma always alternate in tossing. Note that a game is won by the winner tossing a Head. The next toss starts a new game, and the loser goes first in the next game. Hence, Fred and Wilma always alternate in tossing.

As we found above and on Problem Set 2, $P(F_1) = P(F) = \frac{1}{1+q}$ and $P(W_1) = P(W) = \frac{q}{1+q}$.

- (e) Remember that F_n and W_n are complementary events. By the rules of the game, we have that $P(F_{n+1} | F_n) = \frac{q}{1+q}$ and $P(F_{n+1} | W_n) = \frac{1}{1+q}$, and similarly $P(W_{n+1} | F_n) = \frac{1}{1+q}$ and $P(W_{n+1} | W_n) = \frac{q}{1+q}$. Hence, by the law of total probability,

$$\begin{aligned} P(F_{n+1}) &= \frac{q}{1+q}P(F_n) + \frac{1}{1+q}P(W_n) = \frac{1}{1+q} - \frac{1-q}{1+q}P(F_n). \\ P(W_{n+1}) &= \frac{1}{1+q}P(F_n) + \frac{q}{1+q}P(W_n) = \frac{1}{1+q} - \frac{1-q}{1+q}P(W_n). \end{aligned}$$

- (f) Suppose that $P(F_n) = a + b\alpha^n$ for $n \geq 1$. Then we have that

$$a + b\alpha^{n+1} = \frac{1}{1+q} - \frac{1-q}{1+q} \left(a + b\alpha^n \right) \Rightarrow b\alpha^n \left(\alpha + \frac{1-q}{1+q} \right) = \frac{1-2a}{1+q} \text{ for all } n \geq 1.$$

In order for this to hold for all $n > 1$, it must be that $\alpha = -\frac{1-q}{1+q}$ and $a = \frac{1}{2}$. Hence,

$$P(F_n) = \frac{1}{2} + b \left(-\frac{1-q}{1+q} \right)^n$$

and since $P(F_1) = \frac{1}{1+q}$, we get that $b = -\frac{1}{2}$. The calculation for $P(W_n)$ is almost identical, except that the initial condition $P(W_1) = \frac{q}{1+q}$ gives a different value $+\frac{1}{2}$ for b . Hence, we get that

$$P(F_n) = \frac{1}{2} \left[1 - \left(-\frac{1-q}{1+q} \right)^n \right] \quad \text{and} \quad P(W_n) = \frac{1}{2} \left[1 + \left(-\frac{1-q}{1+q} \right)^n \right].$$

Note that $P(F_n) > P(W_n)$ for odd n , and $P(F_n) < P(W_n)$ for even n . Furthermore, since $(1-q)/(1+q) < 1$, the second terms in square brackets get smaller and smaller as n increases, and the game becomes *asymptotically fair*: both Fred's and Wilma's win probabilities converge to $\frac{1}{2}$ regardless of the bias of the coin as long as $0 < p < 1$.

6. [The Sun Also Rises, But Maybe The Cake Doesn't]

Let A , B , and C denote the events that cooks A, B, and C baked any particular cake. Let F denote the event that the cake failed to rise. Asking for the proportion of failures caused by cook A is the same as asking: "Given that a particular cake failed to rise, what is the probability that A baked that cake?" So we want to find $P(A | F)$. We know that $P(A | F) = \frac{P(A \cap F)}{P(F)} = \frac{P(F | A)P(A)}{P(F)}$. We know from the problem setup that $P(F | A) = 0.02$ and $P(A) = 0.5$, $P(F | B) = 0.03$ and $P(B) = 0.3$, $P(F | C) = 0.05$ and $P(C) = 0.2$. We can use the total law of probability to get:
 $P(F) = P(F | A)P(A) + P(F | B)P(B) + P(F | C)P(C) = (0.02)(0.5) + (0.03)(0.3) + (0.05)(0.2) = 0.029$. With this we find that $P(A | F) = \frac{P(F | A)P(A)}{P(F)} = \frac{0.01}{0.029} = 0.3448 \dots$

7. [The (in)famous Monty Hall problem]

- (a) When you first pick, all doors are equally likely, so $P(A) = \frac{1}{3}$.
- (b) For the stay-put strategy, $P(B | A) = 1$, and $P(B | A^c) = 0$. Using Equation 3.4 in book, $P(B) = P(A)P(B | A) + P(A^c)P(B | A^c) = \frac{1}{3}$.
- (c) For the always-switch strategy, $P(B | A) = 0$, $P(B | A^c) = 1$, and so we get $P(B) = \frac{2}{3}$.
 Note that Monty's offer is essentially "Would you rather have what is behind the two doors that you didn't pick (except you don't have to take the goat that has already been revealed home with you), or would you rather stay with your original pick where you have a $\frac{1}{3}$ chance of having picked the right door?" Clearly, the always-switch strategy is best.
- (d) If we pick with equal probabilities $\frac{1}{2}$ one of the two unopened doors, we have that $P(B | A) = P(B | A^c) = \frac{1}{2}$, and so $P(B) = \frac{1}{2}$.
- (e) Picking doors in turn is essentially sampling without replacement, and so it doesn't matter who picks first: both you and your friend have probability $\frac{1}{3}$ of having chosen the door with the car. Note that the probability is $\frac{2}{3}$ that one of the two doors that you two have picked conceals the car, and $\frac{1}{3}$ that the remaining door hides the car. If your friend has been sent back to his seat, you should stay put to maximize your chances of getting the car. Monty's offer is essentially "Would you rather have what's behind both the two doors that have been picked (except that you don't have to take the goat that has been revealed home with you) or would you like to switch to the remaining door which has $\frac{1}{3}$ probability of concealing the car?"