

ECE 313: Problem Set 4: Solutions

Binomial, Geometric, and Poisson Random Variables

1. [The Binomial Random Variable I]

- (a) $p_{\mathbf{X}}(k) = \begin{cases} \binom{8}{k} p^k (1-p)^{8-k}, & 0 \leq k \leq 8, \\ 0, & \text{otherwise.} \end{cases}$
- (b) $P(\text{undetected error}) = p_{ue} = \sum_{k=1}^4 \binom{8}{2k} p^{2k} (1-p)^{8-2k} \approx 2.7832 \times 10^{-5}$
- (c) $p_{\mathbf{Y}}(k) = \begin{cases} \binom{N}{k} p_{ue}^k (1-p_{ue})^{N-k}, & 0 \leq k \leq N, \\ 0, & \text{otherwise.} \end{cases}$
- (d) $E[\mathbf{Y}_n] = p_{ue} \cdot n$. $E[\mathbf{Y}_1] \approx 2.7832 \times 10^{-5}$, $E[\mathbf{Y}_{10}] \approx 2.7832 \times 10^{-4}$,
 $E[\mathbf{Y}_{100}] \approx 2.7832 \times 10^{-3}$, $E[\mathbf{Y}_{1000}] \approx 2.7832 \times 10^{-2}$, $E[\mathbf{Y}_{10000}] \approx 2.7832 \times 10^{-1}$

2. [Geometric Random Variables]

- (a) $p_{\mathbf{Y}}(k) = p(1-p)^k$ for $k \geq 0$.
- (b) $p_{\mathbf{Z}}(k) = p_{ue}(1-p_{ue})^k$ for $k \geq 0$.
- (c) $E[\mathbf{Z}] = \frac{1-p_{ue}}{p_{ue}}$. Therefore, $p_{ue} \leq \frac{1}{10 \times 2^{10}} \approx 9.76 \times 10^{-5}$
Solving $9.76 \times 10^{-5} = \sum_{k=1}^4 \binom{8}{2k} p^{2k} (1-p)^{8-2k}$, $p = .001878$

3. [Poisson Random Variables]

- (a) $E[\mathbf{X}_n] = \lambda = p_{ue} \cdot n$. $E[\mathbf{X}_1] \approx 2.7832 \times 10^{-5}$, $E[\mathbf{X}_{10}] \approx 2.7832 \times 10^{-4}$,
 $E[\mathbf{X}_{100}] \approx 2.7832 \times 10^{-3}$, $E[\mathbf{X}_{1000}] \approx 2.7832 \times 10^{-2}$, $E[\mathbf{X}_{10000}] \approx 2.7832 \times 10^{-1}$
- (b) The results are identical.
- (c) binomial: $p_n = 1 - (1-p_{ue})^n$, $p_1 \approx 2.7832 \times 10^{-5}$, $p_{10} \approx 2.7829 \times 10^{-4}$,
 $p_{100} \approx 2.7794 \times 10^{-3}$, $p_{1000} \approx 2.7449 \times 10^{-2}$, $p_{10000} \approx .2429$.
Poisson: $p_n = 1 - e^{-np_{ue}}$, $p_1 \approx 2.7832 \times 10^{-5}$, $p_{10} \approx 2.7829 \times 10^{-4}$,
 $p_{100} \approx 2.7794 \times 10^{-3}$, $p_{1000} \approx 2.7449 \times 10^{-2}$, $p_{10000} \approx .2429$.

4. [The Binomial Random Variable II]

- (a) The daily demand $\mathbf{X}$ for papers can be modeled as a binomial random variable with parameters (100,0.6). This event occurs with probability $P[\mathbf{X} \geq 50] \approx 0.983$
- (b) If he cannot sell any papers, he loses $\$0.1 \times 50 = \$5$. If he sells all of them, he makes $\$0.25 \times 50 = \$12.50$. If the demand $\mathbf{X}$ is for 50 papers or less, the newsboy sells $\mathbf{X}$ papers and recycles $50 - \mathbf{X}$ papers. Thus, his profit (in cents) is $25\mathbf{X} - 10(50 - \mathbf{X}) = 35\mathbf{X} - 500$. Otherwise, $\mathbf{X} > 50$, and his profit is 1250 regardless of how large $\mathbf{X}$ is. Hence, $\mathbf{Z}$ is related to $\mathbf{X}$ as $\mathbf{Z} = \begin{cases} 35\mathbf{X} - 500, & 0 \leq \mathbf{X} \leq 50, \\ 1250, & \mathbf{X} > 50 \end{cases}$
- (c) $\mathbf{Z}$ takes on values -500, -465, -430, ..., 1215, 1250, 1250, ..., 1250 as $\mathbf{X}$ takes on values 0, 1, 2, ..., 49, 50, 51, ..., 100. Thus, $P[\mathbf{Z} = -500] = P[\mathbf{X} = 0]$, etc. and we can thus evaluate $E[\mathbf{Z}]$ via LOTUS as $E[\mathbf{Z}] = (-500)P[\mathbf{X} = 0] + (-465)P[\mathbf{X} = 1] + \dots + 1215P[\mathbf{X} =$

49] + 1250P[$\mathbf{X} = 50$] + 1250P[$\mathbf{X} = 51$] + 1250P[$\mathbf{X} = 52$] + ... + 1250P[$\mathbf{X} = 100$]. The last 51 terms can be combined to write $E[\mathbf{Z}] = (500)P[\mathbf{X} = 0] + (-465)P[\mathbf{X} = 1] + \dots + 1215P[\mathbf{X} = 49] + 1250P[\mathbf{X} \geq 50] \approx 1248.63$. This is not too surprising since there is a better than 98% chance he sells all his papers and makes 1250 every day. (It would have been surprising if the average profit had been more than 1250, the max profit!)

- (d) The 51st paper provides an extra profit of 25 cents if $\mathbf{X} > 50$ (which occurs with probability ≈ 0.9729) and a loss of 10 cents if $\mathbf{X} \leq 50$ (which occurs with probability ≈ 0.0271). Hence, the average additional profit from the 51st paper is $A(50) \approx 25 \times 0.9729 - 10 \times 0.0271 \approx 24.05$ and the total average profit from 51 papers is approximately $1248.63 + 24.05 = 1272.68$.
- (e) The 52nd paper provides an extra profit of 25 cents if $\mathbf{X} > 51$ (which occurs with probability ≈ 0.958) and a loss of 10 cents if $\mathbf{X} \leq 51$ (which occurs with probability ≈ 0.042). Hence, the average additional profit from the 52nd paper is $A(51) \approx 25 \times 0.95810 - 10 \times 0.042 \approx 23.52$ and the total average profit from 52 papers is approximately $1272.68 + 23.52 = 1296.20$. Notice that on average he makes less additional money (23.52 cents) from the 52nd paper than he does from the 51st (24.05 cents).

5. [The Binomial Random Variable III]

- (a) The number of weeks that your investment doubles in value is a binomial random variable $\mathbb{Y}$ with parameters $(5, \frac{1}{2})$. Since the investment halves in value during the remaining $5 - \mathbb{Y}$ weeks, and each halving cancels one doubling, we have that $\mathbb{X} = 32 \cdot 2^{2\mathbb{Y}-5}$. The possible values of $\mathbb{X}$ are 1, 4, 16, 64, 256, and 1024, corresponding to $\mathbb{Y} = 0, 1, 2, 3, 4, 5$ respectively.
- (b) $P[\mathbb{X} = 1] = P[\mathbb{Y} = 0] = \frac{1}{32}$. $P[\mathbb{X} = 4] = P[\mathbb{Y} = 1] = \frac{5}{32}$.
 $P[\mathbb{X} = 16] = P[\mathbb{Y} = 2] = \frac{10}{32}$. $P[\mathbb{X} = 64] = P[\mathbb{Y} = 3] = \frac{10}{32}$.
 $P[\mathbb{X} = 256] = P[\mathbb{Y} = 4] = \frac{5}{32}$. $P[\mathbb{X} = 1024] = P[\mathbb{Y} = 5] = \frac{1}{32}$.
- (c) $E[\mathbb{X}] = 1 \cdot \frac{1}{32} + 4 \cdot \frac{5}{32} + 16 \cdot \frac{10}{32} + 64 \cdot \frac{10}{32} + 256 \cdot \frac{5}{32} + 1024 \cdot \frac{1}{32} = 97.65625$. The TV commercial *understates* the performance - undoubtedly a first!
- (d) $P[\mathbb{X} < 32] = P[\mathbb{X} = 1] + P[\mathbb{X} = 4] + P[\mathbb{X} = 16] = 1/2$.