

ECE 313: Problem Set 3: Solutions

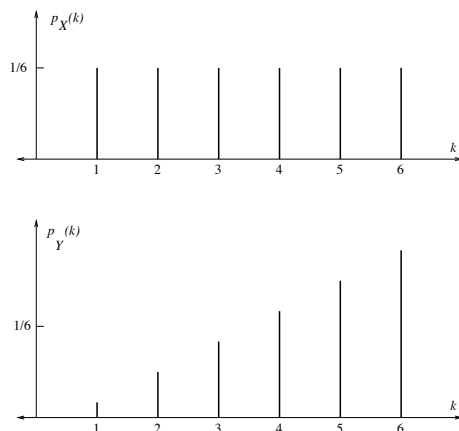
Functions of discrete random variables, and parameter estimation

1. [Mean and standard deviation of a complicated random variable]

- (a) $E[X] = 26.8333\dots$
- (b) $E[X^2] = 1445.1666\dots$
- (c) $\sigma_X = \sqrt{E[X^2] - E[X]^2} = 26.928403$.
- (d) Listing the 36 values of X , we find value 1 listed twice, value 2 listed once, and value 3 is not on the list. Specifically, $X(1,1) = X(2,1) = 1$, and $X(3,2) = 2$. So $p_X(1) = \frac{2}{36} = 0.05555$, $p_X(2) = \frac{1}{36} = 0.027777$, and $p_X(3) = 0$.

2. [Mean and standard deviation of two simple random variables]

- (a) For $1 \leq i \leq 6$, there are six sample points with $X(i, j) = 6$, so $p_X(i) = \frac{6}{36} = \frac{1}{6}$.
- (b) $E[X] = \frac{1+2+3+4+5+6}{6} = 3.5$. To compute σ_X we can first find $E[X^2] = \frac{1+2^2+3^2+4^2+5^2+6^2}{6}$ and then $\sigma_X = \sqrt{E[X^2] - E[X]^2} = 1.707825$.
- (c) $\{Y = 1\} = \{(1,1)\}$, $\{Y = 2\} = \{(2,2), (2,1), (1,2)\}$, $\{Y = 3\} = \{(3,3), (3,2), (3,1), (2,3), (1,3)\}$, and so on. In general, for $1 \leq k \leq 6$, there are $2k - 1$ sample points for which $Y = k$. Thus, $p_Y(k) = \frac{2k-1}{36}$ for $1 \leq k \leq 6$.



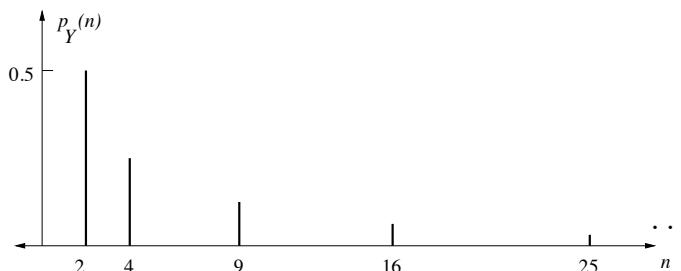
- (d) Using part (c) or reusing the spreadsheet/program from Problem 1, we find $E[Y] = 4.47222$ and $\sigma_Y^2 = 1.40408$.
- (e) $\sigma_Y < \sigma_X$. This is consistent with the sketches. Intuitively, the triangular shape of p_Y is more concentrated than the rectangular shape of p_X .
- (f) Here is one example. Consider a random variable Z with $p_Z(1) = p_Z(6) = 0.5$. Then $E[Z] = \frac{1+6}{2} = 3.5$, $E[Z^2] = \frac{1+36}{2} = 18.5$, and $\sigma_Z = \sqrt{18.5 - (3.5)^2} = 2.5$. In fact, this distribution has the largest variance of any distribution on the set $\{1, 2, 3, 4, 5, 6\}$.

3. [Applying LOTUS for a function of a geometric random variable]

- (a) The possible values of Y are $2, 4, 8, \dots$, and $Y = 2^k$ if and only if $X = k$. Therefore,

$$p_Y(n) = \begin{cases} p(1-p)^{k-1} & \text{if } n = 2^k \text{ for some } k \geq 1 \\ 0 & \text{else} \end{cases}$$

The pmf of Y is pictured for $p = 0.5$.



- (b) $E[Y] = \sum_{k=1}^{\infty} 2^k p(1-p)^{k-1} = 2p \sum_{k=1}^{\infty} [2(1-p)]^{k-1}$. The summation has the form $1 + \theta + \theta^2 + \dots$, with $\theta = 2(1-p)$, which is finite and equal to $\frac{1}{1-\theta}$ if $|\theta| < 1$. Therefore,

$$E[Y] = \begin{cases} \frac{2p}{1-(2(1-p))} = \frac{2p}{2p-1} & \text{if } 0.5 < p \leq 1 \\ +\infty & \text{else.} \end{cases}$$

4. [The mean and variance of the Poisson distribution vs. binomial distribution]

For $n \geq \lambda$ and $p = \lambda/n$, the binomial distribution with parameters n and p has mean $np = \lambda$, and variance $np(1-p) = \lambda(1 - \frac{\lambda}{n})$. As $n \rightarrow \infty$, with λ fixed, these both converge to λ , and λ is the mean and the variance of a Poisson random variable with parameter λ .

5. [Estimation of the parameter of a Poisson random variable]

- (a) Given n , we wish to select λ to maximize $\frac{\lambda^n e^{-\lambda}}{n!}$ over $\lambda \geq 0$. Equivalently, we select λ to maximize $f(\lambda) = \lambda^n e^{-\lambda}$ over $\lambda \geq 0$. If $n = 0$ this function is $e^{-\lambda}$, which is decreasing in λ , so that $\hat{\lambda} = 0$ if $n = 0$. To complete the problem, consider the case $n \geq 1$. In that case, $f'(\lambda) = (n-\lambda)\lambda^{n-1}e^{-\lambda}$. Thus, $f'(\lambda) > 0$ for $\lambda < n$ and $f'(\lambda) < 0$ for $\lambda > n$. Therefore, f is strictly increasing over the interval $[0, n]$ and strictly decreasing over the interval $[n, \infty)$. Thus, the point $\lambda = n$, where $f'(\lambda) = 0$, must be the point at which f is maximized. Thus, $\hat{\lambda} = n$ for $n \geq 1$. Combining the cases $n = 0$ and $n \geq 1$, we see that $\hat{\lambda} = n$ for all values of $n \geq 0$.
- (b) Given m , we wish to select $b \in [0, 10]$ to maximize $\frac{(b^2)^m e^{-b^2}}{m!}$. This is the same as maximizing $f(b^2)$ over $b \in [0, 10]$, where f is the same as in part (a), with n replaced by m . By the analysis in part (a), and the fact b^2 is an increasing function of b over $b \geq 0$, $f(b^2)$ is increasing in b over $[0, \sqrt{m}]$ and it is decreasing in b over $[\sqrt{m}, \infty)$. So if $m \leq 100$, $f(b^2)$ is maximized over $0 \leq b \leq 10$ at $\sqrt{m}$, and if $m > 100$ then $f(b^2)$ is maximized over $0 \leq b \leq 10$ at the right endpoint, 10. Thus, in general, $\hat{b} = \min\{\sqrt{m}, 10\}$.

6. [Parameter estimation for a triangular pmf]

The problem is to maximize $p_X(k)$ with respect to n , for k fixed. As long as $n \geq k$, this probability is strictly positive, and it is zero otherwise. So the maximizing value, $\hat{n}$, must be greater than or equal to k . Now, for $n \geq k$, the probability to be maximized, $\frac{2k}{n(n+1)}$, is strictly decreasing in n . Thus, the maximizing value of n is $\hat{n} = k$.