

ECE 313: Problem Set 2: Solutions

Sets, Events, Axioms of Probability and Their Consequences

1. [Two useful MacLaurin series]

- (a) The two results agree since

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)(n-k)(n-k-1) \cdots 2 \cdot 1}{((n-k)(n-k-1) \cdots 2 \cdot 1)k!} = \frac{n(n-1) \cdots (n-k+1)}{k!}$$

- (b) Using the chain rule,

$$\begin{aligned} g^{(k)}(x) &= (-n)(-n-1)(-n-2) \cdots (-n-k+1)(1-x)^{-n-k}(-1)^k \\ &= n(n+1)(n+2) \cdots (n+k-1)(1-x)^{-n-k} \end{aligned}$$

Thus, $g^{(k)}(0) = n(n+1)(n+2) \cdots (n+k-1) \neq 0$ for all integers $k \geq 0$. Hence, the MacLaurin series for $g(x)$ contains terms of all degrees, and we have that

$$g(x) = (1-x)^{-n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{n(n+1) \cdots (n+k-1)}{k!} x^k.$$

- (c) For $n = 1$, $n(n+1) \cdots (n+k-1) = k!$. For $n = 2$, $n(n+1) \cdots (n+k-1) = (k+1)!$.

$$\text{Hence, } (1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad \text{and} \quad (1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

2. [Subsets of a finite set]

- (a) The subsets of $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ are

Subsets of size 0: $\emptyset$

Subsets of size 1: $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}$

Subsets of size 2: $\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}, \{\omega_2, \omega_4\}, \{\omega_3, \omega_4\}$

Subsets of size 3: $\{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_1, \omega_3, \omega_4\}, \{\omega_2, \omega_3, \omega_4\}$

Subsets of size 4: $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \Omega$

There are $16 = 2^4$ subsets of the set Ω of 4 elements. Only $15 = 2^4 - 1$ of them are *non-empty* subsets.

- (b) OK, OK, geez, some people are never satisfied ...

- (c) By checking our answers in parts (a) and (b), we see that there is: 1 subset of size 0 and 1 subset of size $4 - 0 = 4$, 4 subsets of size 1 and 4 subsets of size $4 - 1 = 3$, and lastly 6 subsets of size 2 with $4 - 2 = 2$. For general n , whenever we form a subset of size k , there is a unique subset of size $n - k$ which is left out. That is, for every subset of size k , there is exactly one corresponding, complementary subset of size $n - k$, so the total number of subsets of size k is the same as the total number of subsets of size $n - k$.
- (d) i. The vector $\{1, 1, \dots, 1\}$ corresponds to Ω . The vector $\{0, 0, \dots, 0\}$ corresponds to $\emptyset$. The n -bit vector with each bit flipped with respect to A corresponds to A^c .
- ii. Since $A \cup B$ contains exactly those elements in either A or B , $z_i = x_i \vee y_i$. Similarly, since $A \cap B$ contains exactly those elements in both A and B , $\omega_i = x_i \wedge y_i$.
- iii. Each bit of the n -bit vector can take on two values independent of the other bits. Therefore, there are 2^n distinct n -bit vectors. As described in the problem setup, there is a *one-to-one correspondence* between n -bit vectors and subsets of Ω , so there are 2^n subsets.

- iv. The questions is: “What are the n -bit vectors with at least one 1”. Of the 2^n subsets of Ω , there is only one empty subset: $\emptyset$. Consequently, there are $2^n - 1$ non-empty subsets.

3. [Unions and Intersections]

- (a) There are $2^7 = 128$ outcomes with $\text{LSB} = 1$, and so $P(A) = \frac{128}{256} = \frac{1}{2}$.
- (b) There are $\binom{8}{5} = \binom{8}{3} = 56$ outcomes in B . Hence $P(B) = \frac{56}{256} = \frac{7}{32}$.
- (c) When the event $A \cap B$ occurs, we know that the 7 more significant bits have exactly 4 ONEs and 3 ZEROes. Hence $|A \cap B| = \binom{7}{4} = \binom{7}{3} = 35$. Thus, $P(A \cap B) = \frac{35}{256}$ and
- $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{128}{256} + \frac{56}{256} - \frac{35}{256} = \frac{149}{256}, \text{ while}$$
- $$P(A \oplus B) = P(A) + P(B) - 2 \cdot P(A \cap B) = \frac{128}{256} + \frac{56}{256} - 2 \cdot \frac{35}{256} = \frac{114}{256} = \frac{57}{128}.$$

4. [A problem on sampling without replacement]

- (a) There are $\binom{2n}{2}$ choices of two shoes from $2n$ of which only n choices yield a pair. Hence,
- $$P(\text{pair}) = \frac{n}{\binom{2n}{2}} = \frac{n}{2n(2n-1)/1 \times 2} = \frac{1}{2n-1}.$$
- More simply, think of choosing the shoes sequentially. Regardless of what the first shoe drawn is, the chance of getting its mate on the second draw is $\frac{1}{2n-1}$. Note that this has value 1 if $n = 1$, which makes perfect sense.
- (b) Any one of the n left shoes can be paired with any one of the n right shoes. So, n^2 of the $\binom{2n}{2}$ choices yield one left shoe and one right shoe in the two drawn, giving that

$$P(\text{one L, one R}) = \frac{n^2}{\binom{2n}{2}} = \frac{n^2}{2n(2n-1)/1 \times 2} = \frac{n}{2n-1}.$$

Again, more simply, regardless of what the first shoe is, the chances of getting a shoe of the opposite footality on the second draw is $\frac{n}{2n-1}$.

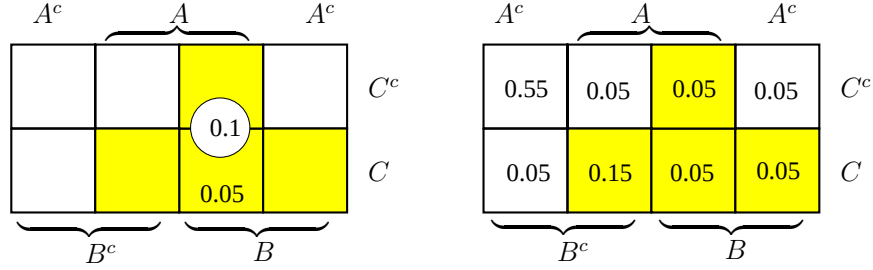
Note that this has value 1 if $n = 1$, which makes perfect sense.

Suppose now that $n \geq 2$ and that you choose 3 shoes at random from the bag.

- (c) Any of the n pairs and any of the other $2n - 2$ shoes form a set of 3 shoes. Hence,
- $$P(\text{pair among three}) = \frac{n(2n-2)}{\binom{2n}{3}} = \frac{n(2n-2)}{2n(2n-1)(2n-2)/1 \times 2 \times 3} = \frac{3}{2n-1}.$$
- Note that this has value 1 when $n = 2$, which makes perfect sense.
- (d) $P(\text{three left shoes}) = \frac{\binom{n}{3}}{\binom{2n}{3}} = \frac{n(n-1)(n-2)}{2n(2n-1)(2n-2)} = \frac{n-2}{8n-4}$. Similarly for all three shoes being right shoes. Hence, $P(\text{one L and one R among three}) = 1 - 2 \times \frac{n-2}{8n-4} = \frac{3n}{4n-2}$.
- Note that this has value 1 when $n = 2$, which makes perfect sense.

5. [The noisy cereal problem]

- (a) The Karnaugh map is as shown in the left hand figure below, with some probabilities marked on it. Note that the shaded region is the event $(A \cap B) \cup (B \cap C) \cup (A \cap C)$.



- (b) Since $A \cap B$ is the disjoint union of $A \cap B \cap C$ and $A \cap B \cap C^c$, we get that $P(A \cap B) = 0.1 = P(A \cap B \cap C) + P(A \cap B \cap C^c) = 0.05 + P(A \cap B \cap C^c)$ giving that $P(A \cap B \cap C^c) = 0.05$. Since $P(AB \cup BC \cup AC) = 0.3 = P(AB) + P(AB^cC) + P(A^cBC)$ while $P(AC) = P(ABC) + P(AB^cC) = 2P(BC) = P(ABC) + P(A^cBC)$, we readily obtain that $P(AB^cC) = 0.15$ and $P(A^cBC) = 0.05$. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$ we have that $P(A^cB^c) = 1 - P(A \cup B) = 0.4$. Since $P(A^cB^cC) = P(C) - P(BC) - P(AB^cC) = 0.05$, we get that $P(\text{cereal snaps, crackles, and pops}) = P(A^c \cap B^c \cap C) = 0.05$. See the figure on the right.
- (c) $P(\text{the sample fails only the snap test}) = P(AB^cC^c) = 0.05$.
 $P(\text{the sample fails only the crackle test}) = P(A^cBC^c) = 0.05$.
 $P(\text{the sample fails only the pop test}) = P(A^cB^cC) = 0.05$.

6. [A problem on countably infinite sample spaces]

- (a) Let q denote $1 - p$ and note that $0 < q < 1$ also. Thus, $P(T^{n-1}H) = q^{n-1}p$ and $P(\Omega) = P\{H, TH, TTH, \dots\} = P\{H\} + P\{TH\} + P\{TTH\} + \dots = p + qp + q^2p + \dots = p[1 + q + q^2 + \dots] = p \frac{1}{1-q} = \frac{p}{p} = 1$. We used the result from Problem 1(b) of Problem Set 1 to sum the series, and will be using this result over and over again in the rest of this problem.
- (b) The event “Fred wins game” is $\{H, TTH, TTTTH, \dots\}$ comprising all the outcomes with an *odd* number of tosses while the event “Wilma wins game” is $\{TH, TTTH, TTTTTH, \dots\}$ comprising all outcomes with an *even* number of tosses.
- (c) $P(\text{Fred wins game}) = P\{H, TTH, TTTTH, \dots\} = P\{H\} + P\{TTH\} + P\{TTTTH\} + \dots = p + q^2p + q^4p + \dots = p[1 + q^2 + q^4 + \dots] = p \frac{1}{1-q^2} = \frac{1}{1+q}$ since $p = 1 - q$ and $1 - q^2 = (1 - q)(1 + q)$.
 $P(\text{Wilma wins game}) = P\{TH, TTTH, TTTTTH, \dots\} = P\{TH\} + P\{TTTH\} + \dots = qp + q^3p + q^5p + \dots = q \cdot p[1 + q^2 + q^4 + \dots] = q \cdot P(\text{Fred wins game}) = \frac{q}{1+q}$.

Obviously, $P(\text{Fred wins game}) + P(\text{Wilma wins game}) = 1$ (and thus $P(\text{tosses go on forever with no winner for all eternity}) = 0$.) Equally obviously, $P(\text{Fred wins game}) = \frac{1}{1+q} > \frac{q}{1+q} = P(\text{Wilma wins game})$ for all p such that $0 < p < 1$. Thus, Fred *always* has an advantage in this game. Note that $P(\text{Fred wins game})$ is only *slightly* larger than $\frac{1}{2}$ when p is close to 0 (and q is close to 1), increases to $\frac{2}{3}$ at $p = \frac{1}{2}$, and approaches 1 as $p \rightarrow 1$.

More generally, if there are k players (Barney? Betty? ...), the win probabilities are

$$\frac{1}{1 + q + q^2 + \dots + q^{k-1}}, \frac{q}{1 + q + q^2 + \dots + q^{k-1}}, \frac{q^2}{1 + q + q^2 + \dots + q^{k-1}}, \dots, \frac{q^{k-1}}{1 + q + q^2 + \dots + q^{k-1}}$$

decreasing by a factor of q from player to player. Note that the k probabilities sum to 1.

- (d) This new game always ends with a HH or a TT . If the number of tosses is even, Wilma wins, while if the number of tosses is odd, Fred wins. Thus, we can *partition* Ω into four disjoint subsets as follows. Let

- A denote the event that there are an *even* number of tosses ending in HH .
- B denote the event that there are an *even* number of tosses ending in TT .
- C denote the event that there are an *odd* number of tosses ending in HH .
- D denote the event that there are an *odd* number of tosses ending in TT .

We can calculate the probabilities of these sets as follows:

$$\begin{aligned} P(A) &= P\{HH, HTHH, HTHTHH, \dots\} = P\{HH\} + P\{HTHH\} + P\{HTHTHH\} + \dots \\ &= p^2 + (pq)p^2 + (pq)^2p^2 + \dots = p^2 \frac{1}{1-pq} = \frac{p^2}{1-pq}. \end{aligned}$$

$$\begin{aligned} P(B) &= P\{TT, THTT, THTHTT, \dots\} = P\{TT\} + P\{THTT\} + P\{THTHTT\} + \dots \\ &= q^2 + (qp)q^2 + (qp)^2q^2 + \dots = q^2 \frac{1}{1-pq} = \frac{q^2}{1-pq}. \end{aligned}$$

$$\begin{aligned} P(C) &= P\{THH, THTHH, THTHTHH, \dots\} = P\{THH\} + P\{THTHH\} + P\{THTHTHH\} + \dots \\ &= qp^2 + (qp)^2p + (qp)^3p + \dots = q \cdot [p^2 + (qp)p^2 + (qp)^2p^2 + \dots] = q \cdot P(A) = \frac{qp^2}{1-pq}. \end{aligned}$$

$$\begin{aligned} P(D) &= P\{HTT, HTHTT, HTHTHTT, \dots\} = P\{HTT\} + P\{HTHTT\} + P\{HTHTHTT\} + \dots \\ &= pq^2 + (pq)^2q + (pq)^3q + \dots = p \cdot [q^2 + (pq)q^2 + (pq)^2q^2 + \dots] = p \cdot P(B) = \frac{pq^2}{1-pq}. \end{aligned}$$

It is easy to verify that

$$P(\Omega) = P(A) + P(B) + P(C) + P(D) = \frac{p^2 + q^2 + qp^2 + pq^2}{1-pq} = \frac{p^2 + q^2 + qp}{1-pq} = 1.$$

- (e) Since Fred starts, and Fred and Wilma alternate turns, all outcomes with an odd number of coin tosses correspond to “Fred wins game.” This is the set $C \cup D$. Similarly, all outcomes with an even number of coin tosses correspond to “Wilma wins game” and this is the set $A \cup B$.
- (f) Note that $P(\text{Wilma wins game}) = P(A) + P(B)$ while $P(\text{Fred wins game}) = P(C) + P(D)$. But, $P(C) = q \cdot P(A) < P(A)$ and $P(D) = p \cdot P(B) < P(B)$ for all values of p , $0 < p < 1$. Hence, $P(\text{Wilma wins game}) > P(\text{Fred wins game})$ for all values of p , $0 < p < 1$, and since the two probabilities sum to 1, it follows that

$$P(\text{Wilma wins game}) > \frac{1}{2} > P(\text{Fred wins game}) \text{ for all values of } p, 0 < p < 1.$$

For this game, $P(\text{Fred wins game}) = \frac{pq}{1-pq}$ is close to 0 if p is nearly 0 or if p is nearly 1. Writing

$P(\text{Fred wins game}) = \frac{pq}{1-pq} = \frac{1}{1-pq} - 1$, we see that the *maximum* value of $P(\text{Fred wins game})$ occurs when $1-pq = 1-p(1-p) = 1-p+p^2$ is minimum. This minimum occurs at $p = \frac{1}{2}$ giving that the *maximum* value of $P(\text{Fred wins game})$ is $\frac{1}{3}$. Of course, this means that the *minimum* value of $P(\text{Wilma wins game})$ is $\frac{2}{3}$. Contrast this to the previous game where with a highly biased coin with p close to 0, Wilma’s chances of winning were only slightly smaller than Fred’s chances of winning!