

ECE 313: Problem Set 1: Solutions

Calculus Review

1. [Geometric, MacLaurin and Taylor series]

- (a) Let $n > 0$ be an integer. Then,

$$(1-x)[1+x+x^2+\cdots+x^{n-1}] = [1+x+x^2+\cdots+x^{n-1}] - [x+x^2+\cdots+x^n] = 1-x^n.$$

If $x \neq 1$, divide both sides by $(1-x)$ to get $1+x+x^2+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$.

- (b) Since $|x| < 1$, $1+x+x^2+\cdots = \lim_{n \rightarrow \infty} 1+x+x^2+\cdots+x^{n-1} = \lim_{n \rightarrow \infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$.
- (c) The first derivative is $f^{(1)}(x) = n(1+x)^{n-1}$, the second is $f^{(2)}(x) = n(n-1)(1+x)^{n-2}$, and so on. The k -th derivative is $f^{(k)}(x) = n(n-1)\cdots(n-k+1)(1+x)^{n-k}$, and finally the n -th derivative is $f^{(n)}(x) = n(n-1)\cdots 2 \cdot 1 = n!$ which is a constant.
- (d) Using the results from part (c) for the first n derivatives, the first $n+1$ terms of the Maclaurin series for $f(x)$ are

$$f(x) = (1+x)^n \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^n \frac{n(n-1)\cdots(n-k+1)}{k!} x^k.$$

- (e) Since $f^{(n)}(x)$ is a constant, $f^{(k)}(x) = 0$ for all $k > n$. Thus, the above *is* the complete Maclaurin series for $f(x)$; there is *no* approximation.

2. [Inverse function]

Sketching f , we see that f is monotonically decreasing, with limit 2 at $-\infty$ and limit 0 at ∞ . In particular, an inverse function exists because f is one-to-one. The range of f is $(0, 2)$, so the inverse g needs only to be defined on that interval. We need $f(g(u)) = u$ for $0 < u < 2$. Will will abbreviate $g(u)$ by the letter g alone, but remember that g is a function of u . So we have $f(g) = u$. Solving this equation for g yields $\frac{2}{e^g+1} = u$, or $\frac{2}{u} - 1 = e^g$, or $g = \ln(\frac{2}{u} - 1)$ for $0 < u < 2$. That is, $g(u) = \ln(\frac{2}{u} - 1)$ for $0 < u < 2$.

3. [Extrema of functions]

- (a) The function $f(x) = x^4(1.001)^{-x^2}$ can be written in the form $x^n \exp(-ax^2)$, where $a = \ln(1.001)$, $n = 4$. The function $x^n \exp(-ax^2)$ has local extrema at the points $x = \pm\infty, 0, \pm\sqrt{n/2a}$. In this case $n = 4$ and therefore the maxima occur at $x = \pm\sqrt{n/2a} = \pm 44.7325$ (as can be verified by differentiating once again and checking the sign of the result at the two points); the points $x = 0, \pm\infty$ represent minima. The maximum function value (at both maxima) is 5.4188×10^5 .
- (b) If $f(x) = x^4(1.001)^{x^2}$, the function converges to infinity as $|x| \rightarrow \infty$. Thus, $f(x)$ is not a maximum value of f for any finite value of x . Incidentally, carrying out the differentiation gives us $x = \pm\sqrt{-2/\ln(1.001)}$ as the other solutions, which is not possible because the quantity within the “ $\sqrt{}$ ” sign is negative.

4. [Some definite integrals]

(a)
$$\int_{-1}^1 |x| dx = \int_{-1}^0 -x dx + \int_0^1 x dx = \left. -\frac{1}{2}x^2 \right|_{-1}^0 + \left. \frac{1}{2}x^2 \right|_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

(b) Substitute $y = (1 - x^2)$. Then

$$\int_0^1 x(1 - x^2)^{11} dx = -\frac{1}{2} \int_1^0 y^{11} dy = \frac{1}{2} \int_0^1 y^{11} dy = \frac{y^{12}}{24} \Big|_0^1 = \frac{1}{24}.$$

(c) Integrate by parts twice:

$$\begin{aligned} \int_0^1 x^2 e^{-x} dx &= -x^2 e^{-x} \Big|_0^1 + \int_0^1 2x e^{-x} dx \\ &= -e^{-1} + \int_0^1 2x e^{-x} dx \\ &= -e^{-1} - 2x e^{-x} \Big|_0^1 + \int_0^1 2e^{-x} dx \\ &= -3e^{-1} + \int_0^1 2e^{-x} dx \\ &= -3e^{-1} + 2(1 - e^{-1}) = 2 - 5e^{-1} \approx 0.1602 \end{aligned}$$

(d) Notice that the integrand is an odd function and therefore it has to integrate to 0 over any interval that is symmetric about the origin.

5. [Derivatives and integrals]

- (a) True as per chain rule.
- (b) Also true as per chain rule.
- (c) False. Chain rule gives $\frac{d}{dx} \exp(f(x^2))$ as $2x \cdot g(x^2) \exp(f(x^2))$.
- (d) False. $\int g(-x) dx = -f(-x) + C$ (minus sign missing).
- (e) False. The antiderivative of $g(x^2/2)$ need not be related to $f(x^2/2)$ at all, as can be seen by differentiating both sides of (v). For example, take $f(x) = \ln(1 + x)$ to see this.
- (f) True only for positive functions $f(x)$.

6. [Double integrals]

- (a) Drawing a three-dimensional picture is not the easiest thing, but it is worth the effort to get an idea of what the function looks like. Applying basic integration techniques and using the fact that $\max(x, y) = y$ if $x \leq y$ and $\max(x, y) = x$ if $x > y$, we get

$$\begin{aligned} \int_0^1 \int_0^1 \max(x, y) dx dy &= \int_0^1 \left(\int_0^y y dx \right) dy + \int_0^1 \left(\int_y^1 x dx \right) dy \\ &= \int_0^1 y^2 dy + \int_0^1 \frac{1 - y^2}{2} dy = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

- (b) Changing to polar coordinates ($x = r \cos \theta, y = r \sin \theta$), the above integral can be reduced to

$$\iint_D (x^2 + y^2)^{-4} dx dy = \int_0^{2\pi} \int_1^\infty r^{-8} \cdot r dr d\theta = 2\pi \int_1^\infty r^{-7} dr = \pi/3,$$

where the two-dimensional differential $dx dy$ transforms to $r dr d\theta$.