University of Illinois Spring 2009

ECE 313: Final Exam

Friday, May 8, 2009, 8:00 a.m. — 11:00 a.m.
149 National Soybean Research Center, 1101 W. Peabody Drive, Urbana

1. **[20 points]** An urn contains five balls numbered consecutively from 1 to 5. Balls are drawn one at a time (*sampling without replacement*) until the 5 ball is drawn. Let $\mathbb{N}$ denote the number of draws required to draw the 5 ball and let $\mathbb{Z}$ denote the sum of the numbers on the balls drawn prior to the 5 ball.

   (a) **[10 points]** Find the pmf of $\mathbb{N}$.

   (b) **[10 points]** Find the conditional expectation $\mathsf{E}[\mathbb{Z} \mid \mathbb{N} = 3]$.

2. **[30 points]** Three coins are in a bag. One has heads on both sides, one has tails on both sides, and the third is a fair coin.

   (a) **[10 points]** First, the instructor takes a coin out of the bag, and flips it twice. What is the probability that the first two coin flips both result in "heads?"

   (b) **[10 points]** Given that the first coin flip results in heads, what is the conditional probability that the second flip of the same coin is also heads?

   (c) **[10 points]** Suppose "heads" shows on each of the first two coin flips, and then a student selects one of the other two coins in the bag and flips it. What is the conditional probability that the coin flip of the student also shows heads?

3. **[16 points]** Suppose $\mathbb{Y} = \mathbb{X}^2$ where $\mathbb{X}$ has pdf $f_{\mathbb{X}}(u) = \begin{cases} 2u, & \text{if } 0 \le u \le 1, \\ 0, & \text{otherwise.} \end{cases}$

   Find the CDF of $\mathbb{Y}$.

4. **[20 points]** Let $(\mathbb{N}(t) : t > 0)$ denote a Poisson process with arrival rate $\lambda > 0$. Remember that this notation means that for each $t > 0$, the number of arrivals in the interval $(0, t]$ is denoted by $\mathbb{N}(t)$.

   (a) **[10 points]** Find $P\{\mathbb{N}(1) = 2, \mathbb{N}(3) = 5\}$. Note that $\mathbb{N}(1)$ and $\mathbb{N}(3)$ are *not* independent. Your answer should depend on λ.

   (b) **[10 points]** Find $P(\mathbb{N}(1) = 2 \mid \mathbb{N}(3) = 5)$.

5. **[20 points]** Suppose $\mathbb{X}$ is a continuous random variable with mean 20 and variance 3. Find the numerical value of $P\{\mathbb{X} < 20 - 1.732\sqrt{3}\}$ (or, nearly equivalently, $P\{\mathbb{X} < 17\}$) in the following two cases:

   (a) **[10 points]** $\mathbb{X}$ is a Gaussian random variable.

   (b) **[10 points]** $\mathbb{X}$ is a uniformly distributed random variable.

6. **[25 points]** $\mathbb{X}$ and $\mathbb{Y}$ are jointly continuous random variables with joint pdf given by

   $$f_{\mathbb{X},\mathbb{Y}}(u, v) = \begin{cases} 2, & \text{if } u \ge 0,\ v \ge 0,\ u + v \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   Find the pdf $f_{\mathbb{Z}}(\alpha)$ of the random variable $\mathbb{Z} = \mathbb{Y}/\mathbb{X}$. To obtain full credit, you must specify the value of $f_{\mathbb{Z}}(\alpha)$ for all α, $-\infty < \alpha < \infty$.

7. **[36 points]** The joint pdf of the random variables $\mathbb{X}$ and $\mathbb{Y}$ has constant value 1 on the triangular region with vertices at $(-1, 0)$, $(0, 1)$, and $(1, 0)$.

   (a) **[8 points]** Find the value of $\mathsf{E}[\mathbb{Y}]$.

   (b) **[6 points]** Find $f_{\mathbb{Y}|\mathbb{X}}(v \mid \frac{\pi}{4})$, the conditional pdf of $\mathbb{Y}$ given that $\mathbb{X} = \pi/4$.

   (c) **[6 points]** Find $\mathsf{E}[\mathbb{Y} \mid \mathbb{X} = \frac{\pi}{4}]$, the conditional mean of $\mathbb{Y}$ given that $\mathbb{X} = \pi/4$.

(d) **[8 points]** Sketch, as a function of u, a graph of the *minimum mean-square error estimator* of $\mathbb{Y}$ given that value of $\mathbb{X}$ is u, for u in the range $u \in (-1, 1)$.

(e) **[8 points]** Sketch a graph of the **linear** *minimum mean-square error estimator* of $\mathbb{Y}$ given that value of $\mathbb{X}$ is u, where $u \in (-1, 1)$.
Hint: Without doing any actual integrations, first deduce that $\mathsf{E}[\mathbb{X}\mathbb{Y}] = \mathsf{E}[\mathbb{X}] = 0$.

8. **[28 points]** Suppose $\mathbb{X}$ and $\mathbb{Y}$ are *zero-mean unit-variance* jointly Gaussian random variables with correlation coefficient $\rho = 0.5$.

(a) **[8 points]** Find $\mathsf{var}(3\mathbb{X} - 2\mathbb{Y})$.

(b) **[10 points]** Find the numerical value of $P\{(3\mathbb{X} - 2\mathbb{Y})^2 \leq 28\}$.

(c) **[10 points]** Find $\mathsf{E}[\mathbb{Y} \mid \mathbb{X} = 3]$.

9. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

TRUE	FALSE	
☐	☐	$P(A \cup B) \geq \max\{P(A), P(B)\}$
☐	☐	$P(A\|B) + P(A\|B^c) = 1$.
☐	☐	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$.
☐	☐	If $P(A) = P(B)$, then $P(A\|B) = P(B\|A)$.

(b) $\mathbb{X}$ is a continuous random variable whose probability density function (pdf) $f_{\mathbb{X}}(u)$ is an *even function*, that is, $f_{\mathbb{X}}(u) = f_{\mathbb{X}}(-u)$ for *all* real numbers u. Let $F_{\mathbb{X}}(u)$ denote the cumulative probability distribution function (CDF) of $\mathbb{X}$, let α denote a *positive* real number, and assume that that $\mathbb{X}$ has finite variance σ^2.

TRUE	FALSE	
☐	☐	$\mathsf{E}[\mathbb{X}] = 0$.
☐	☐	$F_{\mathbb{X}}(-\alpha) \leq \dfrac{\sigma^2}{2\alpha^2}$.
☐	☐	$P\{\mathbb{X}^2 > \alpha^2\} = 2F_{\mathbb{X}}(-\alpha)$.
☐	☐	The pdf of $\mathbb{Y} = \|\mathbb{X}\|$ is $f_{\mathbb{Y}}(v) = 2f_{\mathbb{X}}(v) - 1, 0 \leq v < \infty$.

(c) $\mathbb{X}$ and $\mathbb{Y}$ are random variables with finite and equal variances, that is, $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y}) = \sigma^2 < \infty$. Suppose that $\mathsf{var}(2\mathbb{X} + 3\mathbb{Y} + 4) = \mathsf{var}(3\mathbb{X} - 2\mathbb{Y} + 1)$.

TRUE	FALSE	
☐	☐	$\mathbb{X}$ and $\mathbb{Y}$ are *uncorrelated* random variables.
☐	☐	$\mathsf{var}(2\mathbb{X} + 3\mathbb{Y} + 4) = \mathsf{var}(2\mathbb{X} - 3\mathbb{Y} + 1)$.