

ECE 313: Solutions to Problem Set 8

1. (a) $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \leq u < 1, \\ 1, & u \geq 1. \end{cases}$ is a valid CDF. $P\{|\mathcal{X}| > 0.5\} = 1 - F(0.5) = \frac{3}{4}$.
- (b) $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \leq u \leq 2, \\ 1, & u > 2. \end{cases}$ is *not* a valid CDF since $F(1) = 1 > F(2) = 0$.
- (c) $F(u) = \begin{cases} \frac{1}{2} \exp(2u) & u \leq 0, \\ 1 - \frac{1}{4} \exp(-3u), & u > 0, \end{cases}$ is *not* a valid CDF since it is not right-continuous at 0.
- (d) $F(u) = \begin{cases} \frac{1}{2} \exp(2u) & u < 0, \\ 1 - \frac{1}{4} \exp(-3u), & u \geq 0, \end{cases}$ is a valid CDF.
 $P\{|\mathcal{X}| > 0.5\} = 1 - P\{|\mathcal{X}| \leq 0.5\} = 1 - (F(0.5) - F(-0.5)) = \frac{1}{2} \exp(-1) - \frac{1}{4} \exp(-1.5)$.
2. (a) See Figure 1. It is a mixed random variable.

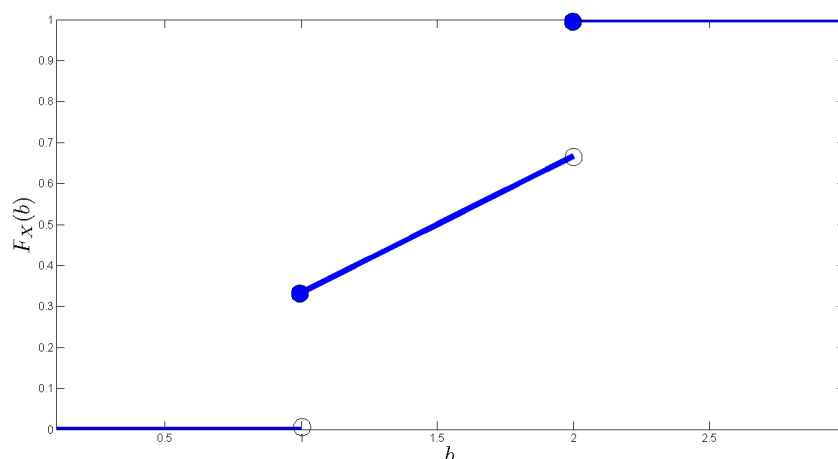


Figure 1: Problem 2(a)

- (b) $E[X] = \int u f_X(u) du + \sum_v v \cdot P\{X = v\} = \int_1^2 u \cdot \frac{1}{3} du + [1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3}] = \frac{3}{2}$
- (c) $P(|X - 1| < 1) = P(0 < X < 2) = F(2^-) - F(0) = \frac{2}{3}$
- (d) $P(|X - 1| < 1 | 1 < X \leq 2) = \frac{P(1 < X < 2)}{P(1 < X \leq 2)} = \frac{F(2^-) - F(1)}{F(2^+) - F(1)} = \frac{2/3 - 1/3}{1 - 1/3} = \frac{1}{2}$
3. (a) See Figure 2.
- (b) $P(|X| < 4) = \int_{-4}^4 \frac{1}{2} e^{-|u|} du = \int_{-4}^0 \frac{1}{2} e^u du + \int_0^4 \frac{1}{2} e^{-u} du = 1 - e^{-4}$, or better still, by symmetry, $P(|X| < 4) = 2 \int_0^4 \frac{1}{2} e^{-u} du = 1 - e^{-4}$.
- (c) Solution to $X^2 + X = 0$ is $X = -1, 0$. Solution to $X^2 + X > 0$ is $X < -1$ or $X > 0$.

$$P(X^2 + X \geq 0) = \int_{-\infty}^{-1} \frac{1}{2} e^u du + \int_0^{\infty} \frac{1}{2} e^{-u} du = \frac{1}{2e} + \frac{1}{2}$$

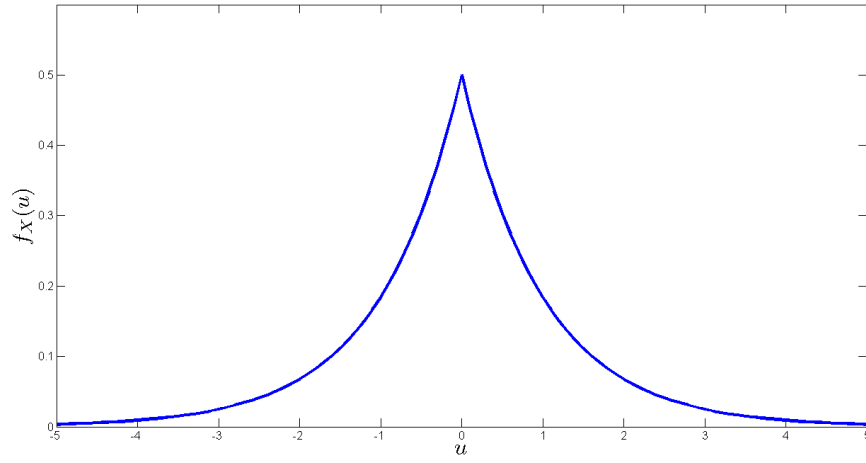


Figure 2: Problem 3(a)

4. (a) If $u \leq 100$, then $F(u) = 0$. If $u \geq 100$, then

$$F(u) = \int_{-\infty}^u f(u) du = \int_{100}^u \frac{100}{u^2} du = 1 - \frac{100}{u}$$

(b) $f(u) = \begin{cases} 1+u & \text{if } -1 \leq u \leq 0 \\ 1-u & \text{if } 0 \leq u \leq 1 \end{cases}$

If $u \leq -1$, then $F(u) = 0$.

If $-1 \leq u \leq 0$, then $F(u) = \int_{-1}^u (1+u) du = \frac{1}{2}u^2 + u + \frac{1}{2}$.

If $0 \leq u \leq 1$, then $F(u) = \int_{-1}^0 (1+u) du + \int_0^u (1-u) du = -\frac{1}{2}u^2 + u + \frac{1}{2}$.

If $u \geq 1$, then $F(u) = 1$.

(c) $F(u) = \int_{-\infty}^u f(u) du$.

If $u \leq -0.1$, then $F(u) = 0$.

If $-0.1 \leq u \leq 0.1$, then $F(u) = \text{area } A = 4(u + 0.1)$.

If $0.1 \leq u \leq 0.5$, then $F(u) = \text{area } B + \text{area } C = \frac{4}{5} + \frac{1}{2}(u - 0.1)$.

If $u \geq 0.5$, then $F(u) = 1$.

All in all,

$$F(u) = \begin{cases} 0 & \text{if } u \leq -0.1 \\ 4(u + 0.1) & \text{if } -0.1 \leq u \leq 0.1 \\ \frac{1}{2}u + \frac{3}{4} & \text{if } 0.1 \leq u \leq 0.5 \\ 1 & \text{if } u \geq 0.5 \end{cases}$$

5. (a) By the properties of a pdf we have

$$1 = \int_{-\infty}^{+\infty} f_X(u) du = c \int_0^3 u du + c \int_3^6 (6-u) du = 9c$$

Hence $c = 1/9$.

(b) We have a continuous random variable, and therefore $F_X(a) = \int_{-\infty}^a f_X(u) du$. Thus

$$\begin{aligned} a < 0 & : F_X(a) = 0 \\ 0 \leq a < 3 & : F_X(a) = \frac{1}{9} \int_0^a u du = \frac{a^2}{18} \\ 3 \leq a < 6 & : F_X(a) = \frac{1}{9} \int_0^3 u du + \frac{1}{9} \int_3^a (6-u) du = \frac{1}{2} - \frac{1}{18}(6-u)^2 \Big|_3^a = 1 - \frac{(6-a)^2}{18} \\ a \geq 6 & : F_X(a) = 1 \end{aligned}$$

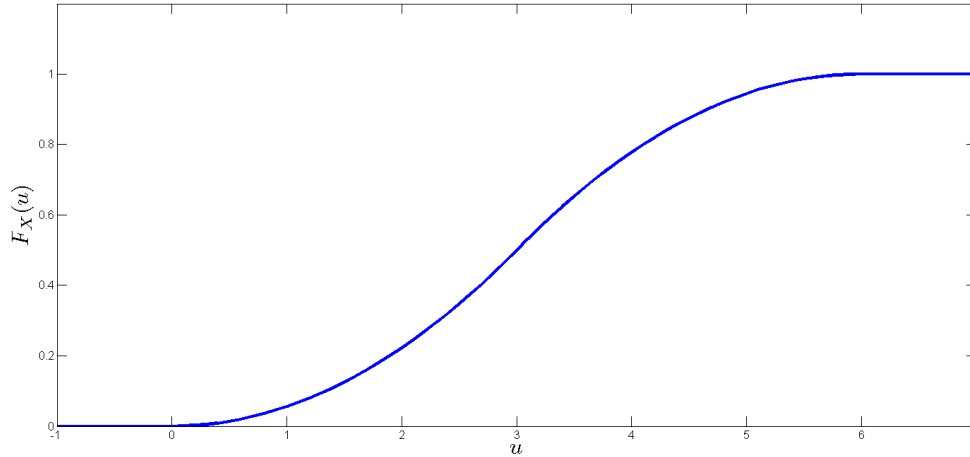


Figure 3: Problem 5(b)

- (c) We have $P(A) = P(X > 3) = 1 - F_X(3) = 0.5$. Similarly $P(B) = P(1.5 \leq X \leq 9) = P(X \leq 9) - P(X < 1.5) = F_X(9) - F_X(1.5^-) = 1 - 0.125 = 0.875$.
- (d) The intersection of A and B is the event $\{3 < X \leq 9\}$ and $P(AB) = P(3 < X \leq 9) = P(X > 3) = 0.5 = P(A) \neq P(A)P(B)$. Hence, the two events are *not* independent.

6. (a) See Figure 4.

To verify that $f_X(u)$ is valid PDF, the following two conditions need to be checked:

$$\begin{aligned} f_X(u) &\geq 0 \quad \forall u \in [0, 2] \\ \int_{-\infty}^{\infty} f_X(u) du &= \text{Area of triangle} = \frac{2 \times 1}{2} = 1 \end{aligned}$$

(b) No.

(c) See Figure 5.

$$\begin{aligned} P\{\text{demand satisfied}\} = P\{X \leq C\} &= \begin{cases} \int_0^C u du & 0 \leq C \leq 1 \\ \int_1^C u du + \int_1^C (2-u) du & 1 \leq C \leq 2 \end{cases} \\ &= \begin{cases} \frac{C^2}{2} & 0 \leq C \leq 1 \\ 2C - \frac{C^2}{2} - 1 & 1 \leq C \leq 2 \end{cases} \end{aligned}$$

If $C = 1 \Rightarrow P\{\text{demand satisfied}\} = \frac{1}{2}$

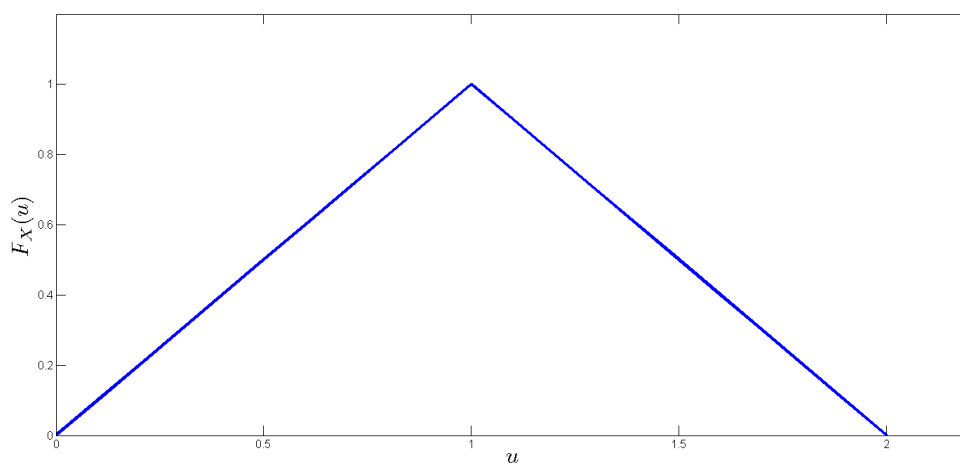


Figure 4: Problem 6(a)

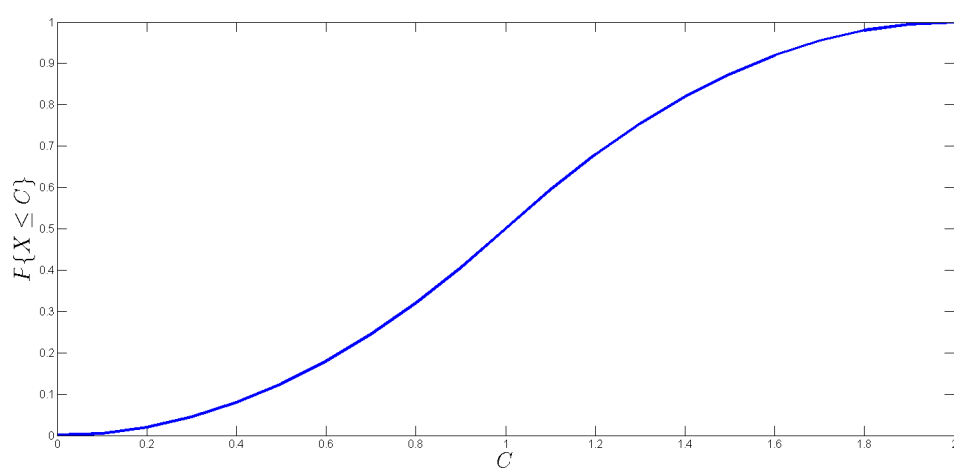


Figure 5: Problem 6(c)

(d)

$$\begin{aligned}
 P\{X > C\} \leq 10^{-1} &\Leftrightarrow 1 - P\{X \leq C\} \leq 10^{-1} \\
 P\{X \leq C\} &\geq 0.9 \\
 2C - \frac{C^2}{2} - 1 &\geq 0.9 \\
 \Rightarrow C &= 1.5528 \text{ gallons}
 \end{aligned}$$

(e) Let Z be a random variable equal to the weekly profit in dollars. Then:

$$Z = g(X) = \begin{cases} 640X, & X < C \\ 640C, & X \geq C \end{cases}$$

If $0 \leq C \leq 1$:

$$\begin{aligned}
 E[Z] &= \int_0^2 g(u) f_X(u) du \\
 &= \int_0^C u \cdot 640u du + \int_C^1 640C \cdot u du + \int_1^2 640C \cdot (2-u) du \\
 &= 640C - \frac{320}{3}C^3
 \end{aligned}$$

If $1 \leq C \leq 2$:

$$\begin{aligned}
 E[Z] &= \int_0^2 g(u) f_X(u) du \\
 &= \int_0^1 640u \cdot u du + \int_1^C 640u \cdot (2-u) du + \int_C^2 640C \cdot (2-u) du \\
 &= \frac{320}{3}C^3 - 640C^2 + 1280C - \frac{640}{3}
 \end{aligned}$$

See sketch below.

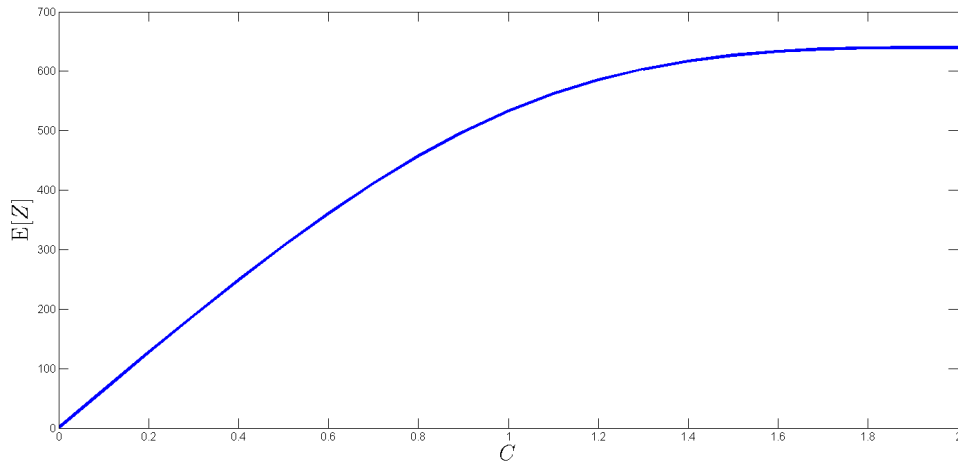


Figure 6: Problem 6(e)

(f) See Figure 7.

$$\begin{aligned}
 \text{Profit} &= E[Z - 10C] \\
 &= E[Z] - 10C \\
 &= \begin{cases} 630C - \frac{320}{3}C^3 & 0 \leq C \leq 1 \\ -\frac{640}{3} + 1270C - 640C^2 + \frac{320}{3}C^3 & 1 \leq C \leq 2 \end{cases}
 \end{aligned}$$

From the graph, it is clear that the value of C that maximizes profit lies between 1 and 2. We set the derivative to 0 to find C .

$$\frac{d}{dC} \left\{ -\frac{640}{3} + 1270C - 640C^2 + \frac{320}{3}C^3 \right\} = 1270 - 1280C + 320C^2 = 0 \Rightarrow C = 1.8232$$

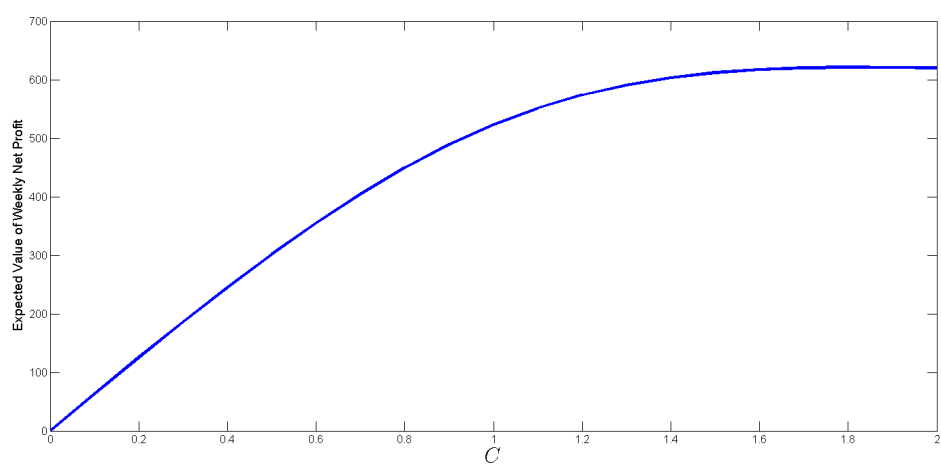


Figure 7: Problem 6(f)