University of Illinois Spring 2008

ECE 313: Solutions to the Final Exam

1. (a) A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

TRUE	$P(A \cup B) \geq \max\{P(A), P(B)\}$. $(A \subset A \cup B, B \subset A \cup B)$
TRUE	$P(A \cap B) \leq \min\{P(A), P(B)\}$. $(A \cap B \subset A, A \cap B \subset B)$
TRUE	If A and B are independent, $P(A\|B) + P(A^c\|B^c) = 1$.
TRUE	$P(A\|B)P(B) + P(A^c\|B^c)P(B^c) = 1 - P(A \oplus B)$.
FALSE	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$. (true if RHS were $P(B)$)
TRUE	If $P(A) = P(B)$, then $P(A\|B) = P(B\|A)$.
FALSE	If $P(A\|B) = P(B\|A)$, then $P(A) = P(B)$. Conclusion does not hold if $P(A\|B) = P(B\|A) = 0$.
TRUE	If $P(A\|B) = P(A)$, then $P(B^c\|A) = 1 - P(B)$. A and B are independent events

(b) $\mathcal{X}$ and $\mathcal{Y}$ are random variables such that $\mathsf{var}(\mathcal{X}) = \mathsf{var}(\mathcal{Y}) = \sigma^2 < \infty$.
$\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = 4\sigma^2 + 9\sigma^2 + 12 \cdot \mathsf{cov}(\mathcal{X}, \mathcal{Y})$
$= \mathsf{var}(3\mathcal{X} - 2\mathcal{Y} + 1) = 9\sigma^2 + 4\sigma^2 - 12 \cdot \mathsf{cov}(\mathcal{X}, \mathcal{Y})$ implies that $\mathsf{cov}(\mathcal{X}, \mathcal{Y}) = 0$.

TRUE	$\mathcal{X}$ and $\mathcal{Y}$ are *uncorrelated* random variables.
FALSE	$\mathcal{X}$ and $\mathcal{Y}$ are *independent* random variables.
TRUE	$\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = \mathsf{var}(2\mathcal{X} - 3\mathcal{Y} + 1)$.
TRUE	$\mathsf{cov}(\mathcal{X} + \mathcal{Y}, \mathcal{X} - \mathcal{Y}) = 0$.

2. At the Democratic National Convention (DNC), Hillary Clinton and Barack Obama have equal numbers of delegates committed to them, and neither candidate can win the nomination on a ballot. In desperation, the DNC decides to have a *series of debates* between the candidiates to decide the Democratic nominee. Hillary wins a debate (event H) with probability p, and Barack wins a debate (event B) with probability $q = 1 - p$. *There are no draws.* The debates continue until one of the candidates wins *two debates in a row* and is declared the Democratic nominee. Successive debates can be regarded as independent trials of an experiment, and $\mathcal{X}$ denotes the total number of debates.

Express the answers to the following questions in terms of p and q, that is, do not write (say) pq as $p(1-p)$ or multiply it out as $p - p^2$. On the other hand, feel free to simplify $p + q = 1$.

(a) For $n > 0$, find the probability that *more than* $2n$ debates occur at the DNC.
Solution: If $2n$ debates have not produced a winner, then the winners *must* have alternated, that is, the result of these $2n$ debates *must* have been either $HBHB \cdots HB$ or $BHBH \cdots BH$. Hence,

$$P\{\mathcal{X} > 2n\} = pqpq \cdots pq + qpqp \cdots qp = 2(pq)^n.$$

Note that this formula does not hold for $n = 0$. (Why not? and what is the value of $P\{\mathcal{X} > 0\}$ anyway?)

(b) For $n \geq 0$, find the probability that *more than* $2n+1$ debates occur at the DNC.
Solution: By the same argument as in part (a) that the winners must alternate, we get that $P\{\mathcal{X} > 2n+1\} = (pq)^n p + (qp)^n q = (pq)^n$ which does hold for $n = 0$.

(c) Find $\mathsf{E}[\mathcal{X}]$. Hint: use the results of parts (a) and (b).

Solution: Since $\mathcal{X}$ is a positive integer-valued random variable, we have that

$$\begin{aligned}
\mathsf{E}[\mathcal{X}] &= \sum_{k=0}^{\infty} P(\mathcal{X} > k) = \sum_{n=0}^{\infty} P\{\mathcal{X} > 2n\} + \sum_{n=0}^{\infty} P\{\mathcal{X} > 2n+1\} \\
&= 1 + \sum_{n=1}^{\infty} P\{\mathcal{X} > 2n\} + 1 + \sum_{n=1}^{\infty} P\{\mathcal{X} > 2n+1\} \\
&= 1 + \sum_{n=1}^{\infty} 2(pq)^n + 1 + \sum_{n=1}^{\infty} (pq)^n = -1 + 3\Big[1 + (pq) + (pq)^2 + \cdots\Big] \\
&= \frac{3}{1-pq} - 1 = \frac{2+pq}{1-pq}.
\end{aligned}$$

Notice that $\mathsf{E}[\mathcal{X}] = 2$ if $p = 0$ or $q = 0$ which makes perfect sense. On the other hand, symmetry indicates that the maximum value of $\mathsf{E}[\mathcal{X}]$ should occur when $p = q = \frac{1}{2}$, and surprisingly, this maximum value is only 3; on average, three or fewer debates can be expected to occur at the DNC!

Alternatively, we can steal the result from part (d) that $P\{\mathcal{X} = 2n+1\} = (pq)^n$ for $n \geq 1$, and use the same argument as in part (d) to deduce that for $n \geq 1$, $P\{\mathcal{X} = 2n\} = P\{\mathcal{X} > 2n-1\} - P\{\mathcal{X} > 2n\} = (pq)^{n-1} - 2(pq)^n$
$= (pq)^{n-1}(1-2pq) = (p^2+q^2)(pq)^{n-1}$. Since $\mathcal{X} \geq 2$, we get

$$\begin{aligned}
\mathsf{E}[\mathcal{X}] &= \sum_{k=2}^{\infty} k \cdot P\{\mathcal{X} = k\} = (p^2+q^2)\sum_{n=1}^{\infty} 2n \cdot (pq)^{n-1} + \sum_{n=1}^{\infty} (2n+1)(pq)^n \\
&= 2(p^2+q^2)\sum_{n=1}^{\infty} n(pq)^{n-1} + 2(pq)\sum_{n=1}^{\infty} n(pq)^{n-1} + \sum_{n=1}^{\infty}(pq)^n \\
&= 2(p^2+q^2+pq)\frac{1}{(1-pq)^2} + \frac{1}{1-pq} - 1 \\
&= \frac{2p^2+2q^2+2pq+1-pq-1+2pq-(pq)^2}{(1-pq)^2} \\
&= \frac{2(p^2+q^2+2pq)-pq-(pq)^2}{(1-pq)^2} = \frac{2(p+q)^2-pq-(pq)^2}{(1-pq)^2} \\
&= \frac{2-pq-(pq)^2}{(1-pq)^2} = \frac{(2+pq)(1-pq)}{(1-pq)^2} = \frac{2+pq}{1-pq} \text{ just as before.}
\end{aligned}$$

(d) Find $P\{\mathcal{X} = 2n+1\}$ and $P\{\mathcal{X} = 2n+1 \big| \mathcal{X} > 2n\}$.

Solution: For integer-valued $\mathcal{X}$, $P(\mathcal{X} = k+1) = P\{\mathcal{X} > k\} - P\{\mathcal{X} > k+1\}$. So, $P\{\mathcal{X} = 2n+1\} = P\{\mathcal{X} > 2n\} - P\{\mathcal{X} > 2n+1\} = 2(pq)^n - (pq)^n = (pq)^n$. Note that since $P\{\mathcal{X} > 0\} = 1 \neq 2(pq)^0$, the result $P\{\mathcal{X} = 2n+1\} = (pq)^n$ does not hold for $n = 0$. For $n = 0$, $P\{\mathcal{X} = 1\} = P\{\mathcal{X} > 0\} - P\{\mathcal{X} > 1\} = 1 - 1 = 0$ as should be obvious if we just think about the problem a bit! From this we get that $P\{\mathcal{X} = 1 \big| \mathcal{X} > 0\} = 0$, while for $n > 0$,

$$\begin{aligned} P\{\mathcal{X} = 2n+1 \big| \mathcal{X} > 2n\} &= \frac{P(\{\mathcal{X} = 2n+1\} \cap \{\mathcal{X} > 2n\})}{P\{\mathcal{X} > 2n\}} \\ &= \frac{P\{\mathcal{X} = 2n+1\}}{P\{\mathcal{X} > 2n\}} = \frac{(pq)^n}{2(pq)^n} = \frac{1}{2}. \end{aligned}$$

Let $\bar{H}$ denote the event that Hillary wins the Democratic nomination. Note that this is not the same as the event H that she wins a debate.

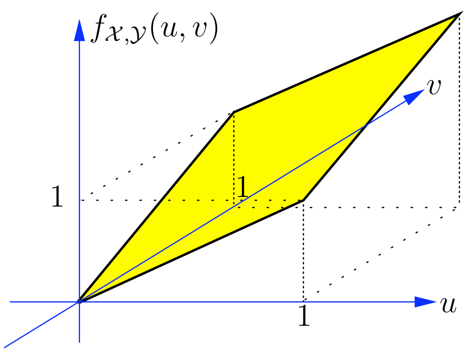
(e) Find $P\{\bar{H} \big| \mathcal{X} = 2n+1\}$.
Solution: From the definition of conditional probability,

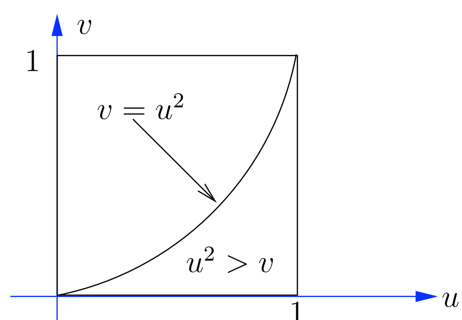
$$P\{\bar{H} \big| \mathcal{X} = 2n+1\} = \frac{P(\bar{H} \cap \{\mathcal{X} = 2n+1\})}{P\{\mathcal{X} = 2n+1\}} = \frac{qpqp\cdots qpp}{P\,(\mathcal{X} = 2n+1)} = \frac{(qp)^n p}{(pq)^n} = p.$$

3. The joint pdf of random variables $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} u+v, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$. Be sure to specify the value of $f_{\mathcal{X}}(u)$ for all real numbers u.
Solution: The pdf surface is a rhombus above the u-v plane as illustrated in the left-hand figure below.
$f_{\mathcal{X}}(u) = \int_{-\infty}^{\infty} f_{\mathcal{X},\mathcal{Y}}(u,v)\,dv = \int_0^1 u + v\,dv = uv + \left.\frac{v^2}{2}\right|_0^1 = u + \frac{1}{2}$, for $0 < u < 1$, and $f_{\mathcal{X}}(u) = 0$ otherwise. It is easily verified that this is a valid pdf.





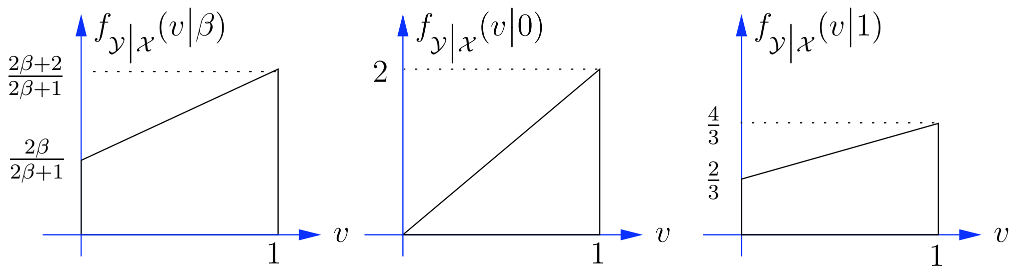
(b) Find the probability that the solutions of the quadratic equation $\alpha^2 + 2\mathcal{X}\alpha + \mathcal{Y} = 0$ are real numbers.
Solution: The solutions to the quadratic equation are $\dfrac{-2\mathcal{X} \pm \sqrt{4\mathcal{X}^2 - 4\mathcal{Y}}}{2}$ and are real numbers if $4\mathcal{X}^2 - 4\mathcal{Y} \geq 0 \Leftrightarrow \mathcal{X}^2 \geq \mathcal{Y}$, that is, if the random point $(\mathcal{X},\mathcal{Y})$

lies below the parabola in the right-hand figure above. Thus, we have that

$$\begin{aligned} P\{\text{real solutions}\} = P\{\mathcal{X}^2 \geq \mathcal{Y}\} &= \int_{u=0}^{1}\int_{v=0}^{u^2}(u+v)dvdu \\ &= \int_{u=0}^{1}\left[uv + \frac{v^2}{2}\right|_{v=0}^{u^2} du = \int_{u=0}^{1}\left[u^3+\frac{u^4}{2}\right]dv \\ &= \left[\frac{u^4}{4}+\frac{u^5}{10}\right|_0^1 = \frac{1}{4}+\frac{1}{10} = \frac{7}{20}. \end{aligned}$$

(c) Find the conditional pdf $f_{\mathcal{Y}|\mathcal{X}}(v|\beta)$ of $\mathcal{Y}$ given that $\mathcal{X} = \beta$, where $0 < \beta < 1$. Be sure to specify the value of $f_{\mathcal{Y}|\mathcal{X}}(v|\beta)$ for all real numbers v.

Solution: $f_{\mathcal{Y}|\mathcal{X}}(v|\beta) = \dfrac{f_{\mathcal{X},\mathcal{Y}}(\beta,v)}{f_{\mathcal{X}}(\beta)} = \dfrac{\beta+v}{\beta+\frac{1}{2}} = \dfrac{2(\beta+v)}{2\beta+1}$ for $0 < v < 1$, and $f_{\mathcal{Y}|\mathcal{X}}(v|\beta) = 0$ otherwise. The pdf is illustrated in the left-hand figure below. Using the formula for the area of a trapezoid, it is easily verified that the area under the curve equals 1.



(d) Find the minimum-mean-square-error (MMSE) estimate of $\mathcal{Y}$ given that $\mathcal{X} = \beta$ where $0 < \beta < 1$.

Solution: The MMSE estimator of $\mathcal{Y}$ given that $\mathcal{X} = \beta$ is the mean of the conditional pdf $f_{\mathcal{Y}|\mathcal{X}}(v|\beta)$, *viz.*,

$$E[\mathcal{X}|\mathcal{Y}=\beta] = \int_0^1 \frac{2v(\beta+v)}{2\beta+1}dv = \frac{1}{2\beta+1}\left[\beta v^2 + \frac{2v^3}{3}\right|_0^1 = \frac{\beta+2/3}{2\beta+1} = \frac{3\beta+2}{6\beta+3}.$$

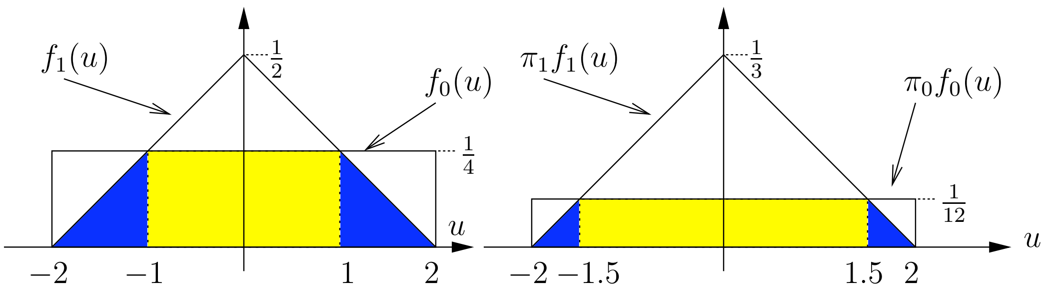
Note that the MMSE estimator is a *nonlinear* function of β, varying from value $\frac{2}{3}$ at $\beta = 0$ to $\frac{5}{9} < \frac{2}{3}$ at $\beta = 1$. Since both $\mathcal{X}$ and $\mathcal{Y}$ are more likely to be close to 1 than to 0, the estimated value of $\mathcal{Y}$ is pretty large even when β, the observed value of $\mathcal{X}$, is close to 0. When β is close to 1, the estimated value of $\mathcal{Y}$ is still large but closer to $\frac{1}{2}$ because the conditional pdf of $\mathcal{Y}$ is much "more uniform" when β is large than when β is small (e.g.. compare the middle and right-hand sketches in the figure on the previous page).

4. Consider the following binary hypothesis testing problem. If hypothesis H_0 is true, the continuous random variable $\mathcal{X} \sim \mathsf{Uniform}(-2,2)$, while if hypothesis H_1 is true, the pdf of $\mathcal{X}$ is $f_1(u) = \begin{cases} \frac{1}{4}(2-|u|), & |u| < 2, \\ 0, & \text{otherwise.} \end{cases}$

(a) The *maximum-likelihood* decision rule can be stated in the form $|\mathcal{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \eta$.

Specify whether x denotes 0 or 1, and find the values of η, the probability of false alarm P_{FA}, and the probability of missed detection P_{MD}.

Solution: The easiest way to solve this problem is to sketch the two pdfs as shown in the left-hand figure below.



It is obvious that the maximum-likelihood decision is in favor of H_1 if $|\mathcal{X}| < 1$, and hence $x = 0$, $\eta = 1$. By inspection, we get that $P_{\text{FA}} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

The graphically-challenged can proceed as follows.

For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{0.25(2 - |u|)}{0.25} = 2 - |u|$.

When $\mathcal{X} = u$ is the observation, the *maximum-likelihood* decision rule decides in favor of H_1 if $\Lambda(u) > 1$. Hence $\Gamma_1 = \{u \,:\, |u| < 1\}$ and $\Gamma_0 = \{u \,:\, 1 < |u| < 2\}$, that is, the ML decision rule is that if $|\mathcal{X}| > 1$, the decision is that H_0 is the true hypothesis. Thus, we have $x = 0$, and $\eta = 1$.

$$P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-1}^{1} \frac{1}{4} du = \frac{1}{2}.$$

$$P_{\text{MD}} = \int_{\Gamma_0} f_1(u)\,du = 2\int_{1}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Bigg|_1^2 = \frac{1}{4}.$$

(b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. What is the error probability $P(E)$ of the maximum-likelihood decision rule?

Solution: The probability of error of the ML decision rule is

$$P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}.$$

(c) The MAP (also known as the minimum-error-probability or Bayesian) decision rule can be stated in the form $|\mathcal{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \xi$. Specify whether x denotes 0 or 1, and find the values of ξ and the error probability $P(E)$.

Solution: Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure above, we easily see that the MAP decision is in favor of H_1 if $|\mathcal{X}| < 1.5$, and hence $x = 0$, $\xi = 1.5$. By inspection, we get that $\pi_0 P_{\text{FA}} = 3 \times \frac{1}{12} = \frac{1}{4} = \frac{1}{3} \times \frac{3}{4}$ while $\pi_1 P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{12}\right) = \frac{1}{24} = \frac{2}{3} \times \frac{1}{16}$, that is, $P_{\text{FA}} = \frac{3}{4}$, and $P_{\text{MD}} = \frac{1}{16}$.

$P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}$, where $\frac{1}{3}$ is the error probability of the ML rule (with the same *a priori* probabilities) that we found in part (c).

Without using any graphical aids, we have that when $\mathcal{X} = u$ is the observation, the MAP decision rule decides in favor of H_1 if $\Lambda(u) = 2 - |u| > \pi_0/\pi_1 = 1/2$. Hence, $\Gamma_1 = \left\{u \,:\, |u| < \frac{3}{2}\right\}$ and $\Gamma_0 = \left\{u \,:\, \frac{3}{2} < |u| < 2\right\}$ for the MAP decision rule. Once again, $x = 0$ while $\xi = \frac{3}{2}$. We get $P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\,du = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4}du = \frac{3}{4}$.

$$P_{\text{MD}} = \int_{\Gamma_0} f_1(u)\,du = 2\int_{\frac{3}{2}}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Big|_{\frac{3}{2}}^{2} = \frac{1}{16}.$$

Hence, $P(E) = \pi_0 \cdot P_{\text{FA}} + \pi_1 \cdot P_{\text{MD}} = \dfrac{1}{3} \times \dfrac{3}{4} + \dfrac{2}{3} \times \dfrac{1}{16} = \dfrac{1}{4} + \dfrac{1}{24} = \dfrac{7}{24} < \dfrac{1}{3}$.

(d) For what range (if any) of values of π_0, does the MAP decision rule always choose H_0?

Solution: The MAP decision rule compares the likelihood ratio $\Lambda(u) = 2 - |u|$ and decides in favor of H_0 if $\Lambda(u) < \pi_0/\pi_1$. Since $\Lambda(u)$ takes on values in $(0, 2]$, it cannot exceed π_0/π_1 if $\pi_0 > \frac{2}{3}$. Thus, the MAP decision rule always decides in favor of H_0 if $\pi_0 > 2/3$, and achieves average error probability $P(E) = \pi_1 < 1/3$.

(e) For what range (if any) of values of π_0, does the MAP decision rule always choose H_1?

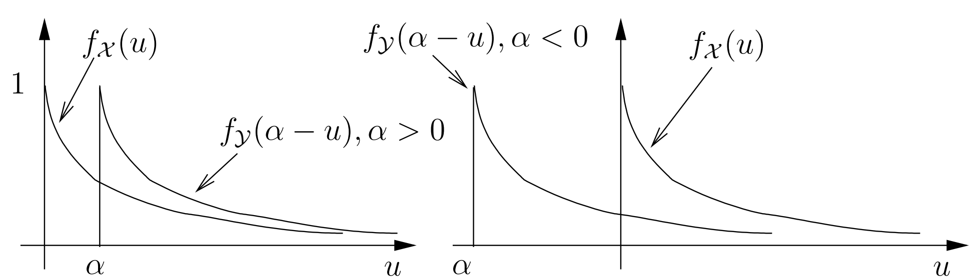
Solution: On the other hand, there is *no* value of $\pi_0, 0 < \pi_0 < 1$ for which the MAP rule *always* decide in favor of H_1, because no matter how large π_1 is, the ratio $\pi_0/\pi_1 > 0$. Thus, for *some* values of $\mathcal{X}$, the likelihood ratio will be smaller than the threshold, and the decision will be in favor of H_0.

5. $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables with pdfs as specified below:

$$f_{\mathcal{X}}(u) = \begin{cases} \exp(-u), & u > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\mathcal{Y}}(v) = \begin{cases} \exp(v), & v < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{\mathcal{Z}}(\alpha)$, the pdf of the random variable $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all real numbers α.

Solution: Since $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables, $f_{\mathcal{Z}} = f_{\mathcal{X}} \star f_{\mathcal{Y}}$, that is, $f_{\mathcal{Z}}(\alpha) = \displaystyle\int_{-\infty}^{\infty} f_{\mathcal{X}}(u) f_{\mathcal{Y}}(\alpha - u)\,du = \int_{0}^{\infty} \exp(-u) f_{\mathcal{Y}}(\alpha - u)\,du$ since $f_{\mathcal{X}}(u) = 0$ for $u < 0$. Now, $f_{\mathcal{Y}}(\alpha - u) = 0$ if $\alpha - u > 0$, that is, if $u < \alpha$. As illustrated in the figure below, we have to consider two cases.



- If $\alpha > 0$, then the integrand is 0 for $0 < u < \alpha$, and we get
$$f_{\mathcal{Z}}(\alpha) = \int_{\alpha}^{\infty} \exp(-u)\exp(\alpha - u)\,du = \exp(\alpha)\frac{1}{2}\exp(-2u)\Big|_{\alpha}^{\infty} = \frac{1}{2}\exp(-\alpha).$$
- If $\alpha < 0$, then the integrand is nonzero for $0 < u < \infty$, and we get
$$f_{\mathcal{Z}}(\alpha) = \int_{0}^{\infty} \exp(-u)\exp(\alpha - u)\,du = \exp(\alpha)\frac{1}{2}\exp(-2u)\Big|_{0}^{\infty} = \frac{1}{2}\exp(\alpha).$$

We can combine these two cases and write $f_{\mathcal{Z}}(\alpha) = \frac{1}{2}\exp(-|\alpha|), -\infty < \alpha < \infty$.

6. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables with means 0 and 2 respectively, variances 4 and 16 respectively, and correlation coefficient $\frac{1}{2}$. Let $\mathcal{W} = -\mathcal{X} + b\mathcal{Y}$ and $\mathcal{Z} = 7\mathcal{X} - \mathcal{Y}$ where b is a number whose value you will determine below.

   (a) For what value(s) of b does $\mathsf{E}[\mathcal{W}]$ equal 0?
   Solution: $\mathsf{E}[\mathcal{W}] = -\mathsf{E}[\mathcal{X}] + b\mathsf{E}[\mathcal{Y}] = 0 + 2b = 0 \Rightarrow b = 0$.

   (b) [**10 points**] For what value(s) of b does $\mathsf{var}(\mathcal{W})$ equal 3?
   Solution: $3 = \mathsf{var}(\mathcal{W}) = \mathsf{var}(\mathcal{X}) + b^2\mathsf{var}(\mathcal{Y}) - 2b\mathsf{cov}(\mathcal{X},\mathcal{Y}) = 4 + 16b^2 - 2b\frac{1}{2}2\cdot 4$ implies that $16b^2 - 8b + 1 = 0 \Rightarrow (4b-1)^2 = 0 \Rightarrow b = \frac{1}{4}$.

   (c) For what value(s) of b are $\mathcal{W}$ and $\mathcal{Z}$ independent random variables?
   Solution: Since $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables, $\mathcal{W}$ and $\mathcal{Z}$ are also jointly Gaussian random variables, and hence independent if $\mathsf{cov}(\mathcal{W},\mathcal{Z}) = 0$. But,
$$\begin{aligned}
\mathsf{cov}(\mathcal{W},\mathcal{Z}) &= \mathsf{cov}(-\mathcal{X} + b\mathcal{Y}, 7\mathcal{X} - \mathcal{Y}) \\
&= -7\cdot\mathsf{cov}(\mathcal{X},\mathcal{X}) + 7b\cdot\mathsf{cov}(\mathcal{Y},\mathcal{X}) + \mathsf{cov}(\mathcal{X},\mathcal{Y}) - b\cdot\mathsf{cov}(\mathcal{Y},\mathcal{Y}) \\
&= -7\cdot 4 + 7b\cdot 4 + 4 - b\cdot 16 \\
&= 12b - 24
\end{aligned}$$
   and we see that $\mathsf{cov}(\mathcal{W},\mathcal{Z}) = 0$ when $b = 2$.