

ECE 313: Conflict Final Exam Solutions

1. (a) A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

TRUE $P(A \cup B) \geq \max\{P(A), P(B)\}$. ($A \subset A \cup B, B \subset A \cup B$)
 TRUE $P(A \cap B) \leq \min\{P(A), P(B)\}$. ($A \cap B \subset A, A \cap B \subset B$)
 TRUE If A and B are independent, $P(A|B) + P(A^c|B^c) = 1$.
 TRUE $P(A|B)P(B) + P(A^c|B^c)P(B^c) = 1 - P(A \oplus B)$.
 FALSE $P(A|B)P(B) + P(A^c|B)P(B) = P(A)$. (true if RHS were $P(B)$)
 TRUE If $P(A) = P(B)$, then $P(A|B) = P(B|A)$.
 FALSE If $P(A|B) = P(B|A)$, then $P(A) = P(B)$. Conclusion does not hold if $P(AB) = 0$.
 TRUE If $P(A|B) = P(A)$, then $P(B^c|A) = 1 - P(B)$. A and B are independent events

- (b) $\mathcal{X}$ and $\mathcal{Y}$ are random variables such that $\text{var}(\mathcal{X}) = \text{var}(\mathcal{Y}) = \sigma^2 < \infty$.
 $\text{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = 4\sigma^2 + 9\sigma^2 + 12 \cdot \text{cov}(\mathcal{X}, \mathcal{Y})$
 $= \text{var}(3\mathcal{X} - 2\mathcal{Y} + 1) = 9\sigma^2 + 4\sigma^2 - 12 \cdot \text{cov}(\mathcal{X}, \mathcal{Y})$ implies that $\text{cov}(\mathcal{X}, \mathcal{Y}) = 0$.
 TRUE $\mathcal{X}$ and $\mathcal{Y}$ are *uncorrelated* random variables.
 FALSE $\mathcal{X}$ and $\mathcal{Y}$ are *independent* random variables.
 TRUE $\text{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = \text{var}(2\mathcal{X} - 3\mathcal{Y} + 1)$.
 TRUE $\text{cov}(\mathcal{X} + \mathcal{Y}, \mathcal{X} - \mathcal{Y}) = 0$.

2. At the Democratic National Convention (DNC), Hillary Clinton and Barack Obama have equal numbers of delegates committed to them, and neither candidate can win the nomination on a ballot. In desperation, the DNC decides to have a *series of debates* between the candidates to decide the Democratic nominee. Hillary wins a debate (event H) with probability p , and Barack wins a debate (event B) with probability $q = 1 - p$. *There are no draws*. The debates continue until one of the candidates has won *two more* debates than the other candidate and is declared the Democratic nominee. Successive debates can be regarded as independent trials of an experiment, and $\mathcal{X}$ denotes the total number of debates at the DNC.

Express the answers to the following questions in terms of p and q , that is, do not write (say) pq as $p(1 - p)$ or multiply it out as $p - p^2$. On the other hand, feel free to simplify $p + q = 1$.

- (a) Find $P\{\mathcal{X} = 2\}$, $P\{\mathcal{X} = 3\}$, $P\{\mathcal{X} = 4\}$, and $P\{\mathcal{X} = 5\}$.

Solution: Suppose that the loser won m debates. Then, the winner won $m + 2$ debates and so the total number of debates is $2m + 2$ which is an even number. Therefore, $P\{\mathcal{X} = 3\} = P\{\mathcal{X} = 5\} = 0$. On the other hand, $P\{\mathcal{X} = 2\} = P\{HH \cup BB\} = p^2 + q^2$, while $P\{\mathcal{X} = 4\} = P\{(HB \cup BH)HH \cup (HB \cup BH)BB\} = (pq + qp)p^2 + (pq + qp)q^2 = (2pq)(p^2 + q^2)$.

- (b) More generally, what is $P\{\mathcal{X} = 2n\}$ for $n \geq 1$?

Solution: The last two debates must have resulted in BB or HH . For the previous debates, the $(2k - 1)$ -th and the $(2k)$ -th debates must have resulted in HB or BH for $k = 1, 2, \dots, n - 1$. Hence,

$$P\{\mathcal{X} = 2n\} = (p^2 + q^2)(2pq)^{n-1} = (1 - 2pq)(2pq)^{n-1}.$$

Sports fans should compare this to the game of tennis when the score has reached deuce: *pairs* of points are played until one player wins both points in a pair, thereby ending the game.

- (c) Find $E[\mathcal{X}]$.

Solution: Note that $\mathcal{X} = 2\mathcal{Y}$ where $\mathcal{Y}$ is a *geometric* random variable with parameter $1 - 2pq$. So $E[\mathcal{X}] = 2E[\mathcal{Y}] = \frac{2}{1 - 2pq} = \frac{2}{p^2 + q^2}$.

Notice that $E[\mathcal{X}] = 2$ if $p = 0$ or $q = 0$ which makes perfect sense. On the other hand, symmetry indicates that the maximum value of $E[\mathcal{X}]$ should occur when $p = q = \frac{1}{2}$, and surprisingly, this maximum value is only 4; on average, four or fewer debates can be expected to occur at the DNC, and four or fewer points played between equally matched players if the score reaches deuce!

- (d) Find $P\{\bar{H}|\mathcal{X} = 2n\}$ and $P(\bar{H})$.

Solution: $P\{\bar{H}|\mathcal{X} = 2n\} = \frac{P\{\bar{H} \cap \mathcal{X} = 2n\}}{P\{\mathcal{X} = 2n\}} = \frac{p^2(2pq)^{n-1}}{(p^2 + q^2)(2pq)^{n-1}} = \frac{p^2}{p^2 + q^2}$.

Since $P\{\bar{H}|\mathcal{X} = 2n\}$ does not depend on n , it follows that

$$P\{\bar{H}\} = P\{\bar{H}|\mathcal{X} = 2n\} = \frac{p^2}{p^2 + q^2} = \frac{p^2}{1 - 2pq}$$

- (e) Find $P\{\mathcal{X} = 2n|\bar{H}\}$.

Solution: Using Bayes' rule, we have

$$\begin{aligned} P\{\mathcal{X} = 2n|\bar{H}\} &= \frac{P\{\bar{H}|\mathcal{X} = 2n\}P\{\mathcal{X} = 2n\}}{P\{\bar{H}\}} = \frac{\left(\frac{p^2}{p^2 + q^2}\right)(p^2 + q^2)(2pq)^{n-1}}{\frac{p^2}{1 - 2pq}} \\ &= (1 - 2pq)(2pq)^{n-1} = P\{\mathcal{X} = 2n\}. \end{aligned}$$

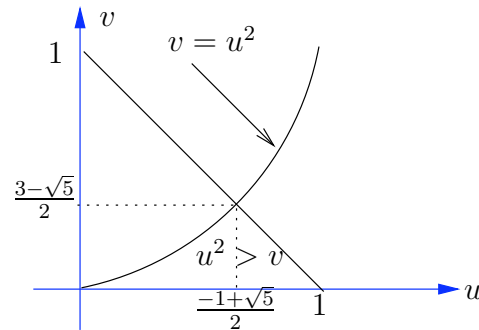
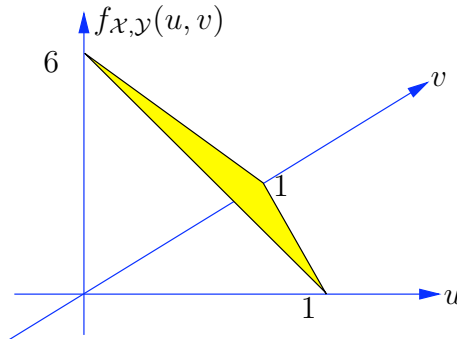
This can also be found by noticing, from part (d), that the two events are independent. Thus it follows that $P\{\mathcal{X} = 2n|\bar{H}\} = P\{\mathcal{X} = 2n\} = (1 - 2pq)(2pq)^{n-1}$.

3. The joint pdf of random variables $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f_{\mathcal{X},\mathcal{Y}}(u, v) = \begin{cases} 6(1 - u - v), & 0 < u < 1, 0 < v < 1, u + v < 1 \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$. Be sure to specify the value of $f_{\mathcal{X}}(u)$ for all real numbers u .

Solution: The pdf surface is an isosceles triangle above the u - v plane as shown in the left-hand figure below.



For $0 < u < 1$, $f_{\mathcal{X}}(u) = \int_{v=0}^{1-u} 6(1-u-v) dv = 6 \left[v - uv - \frac{v^2}{2} \right]_{v=0}^{1-u} = 3(1-u)^2$.

Hence, $f_{\mathcal{X}}(u) = \begin{cases} 3(1-u)^2, & 0 < u < 1, \\ 0, & \text{otherwise.} \end{cases}$

- (b) Find the probability that the solutions of the quadratic equation $\alpha^2 + 2\mathcal{X}\alpha + \mathcal{Y} = 0$ are real numbers.

Solution: The solutions to the quadratic equation are $\frac{-2\mathcal{X} \pm \sqrt{4\mathcal{X}^2 - 4\mathcal{Y}}}{2}$ and are real numbers if $4\mathcal{X}^2 - 4\mathcal{Y} \geq 0 \Leftrightarrow \mathcal{X}^2 \geq \mathcal{Y}$, that is, if the random point $(\mathcal{X}, \mathcal{Y})$ lies below the parabola in the right-hand figure on the previous page. The curves $v = u^2$ and $v = 1 - u$ intersect at $(u, v) = (\frac{-1+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}) = (a, b)$ in the figure. Thus, we have that

$$\begin{aligned} P\{\text{real solutions}\} &= P\{\mathcal{X}^2 \geq \mathcal{Y}\} = \int_{v=0}^b \int_{u=\sqrt{v}}^{1-v} 6(1-u-v) du dv \\ &= \int_{v=0}^b \left[6(1-v)u - 3u^2 \right]_{u=\sqrt{v}}^{1-v} dv \\ &= \int_{v=0}^b 3(1-v)^2 - 6(1-v)\sqrt{v} + 3v dv \\ &= \left[-(1-v)^3 - 4v^{3/2} + \frac{12v^{5/2}}{5} + \frac{3v^2}{2} \right]_0^b \\ &= b^3 - \frac{3}{2}b^2 + 3b + b\sqrt{b} \left[\frac{12b}{5} - 4 \right]. \end{aligned}$$

Alternatively,

$$\begin{aligned} P\{\text{real solutions}\} &= P(\mathcal{Y} \leq \mathcal{X}^2) = \int_{u=0}^1 \int_{v=0}^{\min(u^2, 1-u)} 6(1-u-v) dv du \\ &= \int_{u=0}^a \int_{v=0}^{u^2} 6(1-u-v) dv du + \int_{u=a}^1 \int_{v=0}^{1-u} 6(1-u-v) dv du \\ &= 6 \int_{u=0}^a \left[v - uv - \frac{v^2}{2} \right]_{v=0}^{u^2} du + 6 \int_{u=a}^1 \left[v(1-u) - \frac{v^2}{2} \right]_{v=0}^{1-u} du \\ &= 6 \int_{u=0}^a \left(u^2 - u^3 - \frac{u^4}{2} \right) du + 6 \int_{u=a}^1 \left(\frac{(1-u)^2}{2} \right) du \\ &= 6 \int_{u=0}^a \left(u^2 - u^3 - \frac{u^4}{2} \right) du + 6 \int_{\tilde{u}=0}^{1-a} \left(\frac{\tilde{u}^2}{2} \right) d\tilde{u} \\ &= 6 \left[\frac{u^3}{3} - \frac{u^4}{4} - \frac{u^5}{10} \right]_{u=0}^a + [\tilde{u}^3]_{\tilde{u}=0}^{1-a} \\ &= 6 \left(\frac{a^3}{3} - \frac{a^4}{4} - \frac{a^5}{10} \right) + (1-a)^3 \end{aligned}$$

Although the two solutions look very different, it is possible to transform one into the other by judicious use of the identities $b = 1 - a$ and $b = a^2$.

- (c) Find the conditional pdf $f_{\mathcal{Y}|\mathcal{X}}(v|\beta)$ of $\mathcal{Y}$ given that $\mathcal{X} = \beta$, where $0 < \beta < 1$. Be sure to specify the value of $f_{\mathcal{Y}|\mathcal{X}}(v|\beta)$ for all real numbers v .

Solution: For $0 < \beta < 1$, $f_{\mathcal{Y}|\mathcal{X}}(v|\beta) = \frac{f_{\mathcal{X},\mathcal{Y}}(u, \beta)}{f_{\mathcal{X}}(\beta)} = \frac{6(1 - \beta - v)}{3(1 - \beta)^2} = \frac{2(1 - \beta - v)}{(1 - \beta)^2}$.

So we have $f_{\mathcal{Y}|\mathcal{X}}(v|\beta) = \begin{cases} \frac{2(1 - \beta - v)}{(1 - \beta)^2}, & 0 < v < 1 - \beta, \\ 0, & \text{otherwise.} \end{cases}$

Notice that the conditional pdf is a linear function of v , decreasing from $\frac{2}{1 - \beta}$ at $v = 0$ to 0 at $v = 1 - \beta$.

- (d) Find the minimum-mean-square-error (MMSE) estimate of $\mathcal{Y}$ given that $\mathcal{X} = \beta$ where $0 < \beta < 1$.

Solution: The MMSE estimator is given by $E[\mathcal{Y}|\mathcal{X} = \beta]$. So we have:

$$\begin{aligned} E[\mathcal{Y}|\mathcal{X} = \beta] &= \int_{v=0}^{1-\beta} v f_{\mathcal{Y}|\mathcal{X}}(v|\beta) dv = \frac{2}{(1 - \beta)^2} \int_{v=0}^{1-\beta} v(1 - \beta - v) dv \\ &= \frac{2}{(1 - \beta)^2} \left[(1 - \beta) \frac{v^2}{2} - \frac{v^3}{3} \right]_{v=0}^{1-\beta} = \frac{2}{(1 - \beta)^2} \left[\frac{1 - \beta}{2} - \frac{1}{3} \right] \\ &= \frac{1}{3}(1 - \beta). \end{aligned}$$

Note that this is a *linear* function of the observed value of $\mathcal{X}$, and so must be the same as the *linear MMSE estimator* of $\mathcal{Y}$ in terms of $\mathcal{X}$.

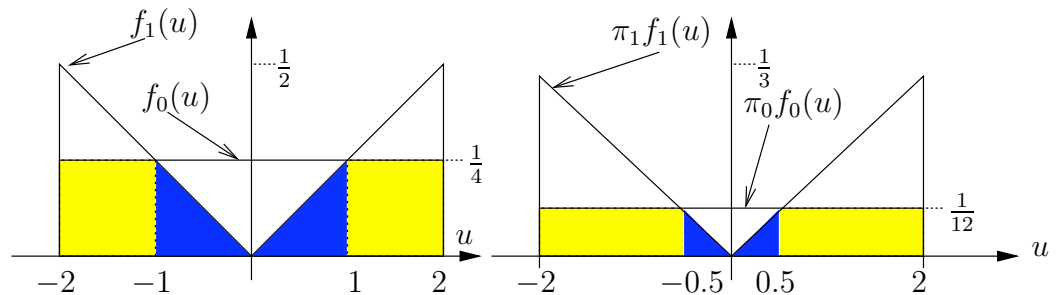
4. Consider the following binary hypothesis testing problem. If hypothesis H_0 is true, the continuous random variable $\mathcal{X} \sim U(-2, 2)$, while if hypothesis H_1 is true, the pdf

of $\mathcal{X}$ is $f_1(u) = \begin{cases} \frac{|u|}{4}, & |u| < 2, \\ 0, & \text{otherwise.} \end{cases}$

- (a) The *maximum-likelihood* decision rule can be stated in the form $|\mathcal{X}| \underset{H_{1-x}}{\overset{H_x}{\geq}} \eta$.

Specify whether x denotes 0 or 1, and find the values of η , the probability of false alarm P_{FA} , and the probability of missed detection P_{MD} .

Solution: The easiest way to solve this problem is to sketch the two pdfs as shown in the left-hand figure below.



It is obvious that the maximum-likelihood decision is in favor of H_1 if $|\mathcal{X}| > 1$, and hence $x = 1$, $\eta = 1$. By inspection, we get that $P_{FA} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{MD} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

The graphically-challenged can proceed as follows.

For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = |u|$.

When $\mathcal{X} = u$ is the observation, the maximum-likelihood decision rule decides in favor of H_1 if $\Lambda(u) > 1$, that is, $|\mathcal{X}| > 1$, the decision is that H_0 is the true hypothesis. Thus, we have $x = 1$, and $\eta = 1$. Note that $\Gamma_1 = \{u : |u| > 1\}$ and $\Gamma_0 = \{u : |u| < 1\}$,

$$P_{FA} = \int_{\Gamma_1} f_0(u) du = 2 \int_1^2 \frac{1}{4} du = \frac{1}{2}.$$

$$P_{MD} = \int_{\Gamma_0} f_1(u) du = \int_{-1}^1 \frac{1}{4} |u| du = \frac{1}{2} \int_0^1 u du = \frac{1}{4}.$$

- (b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. What is the error probability $P(E)$ of the maximum-likelihood decision rule?

Solution: $P(E) = \pi_0 P_{FA} + \pi_1 P_{MD} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}$.

- (c) The MAP (also known as the minimum-error-probability or Bayesian) decision

rule can be stated in the form $\begin{matrix} H_x \\ | \mathcal{X} | \geq \xi \\ H_{1-x} \end{matrix}$. Specify whether x denotes 0 or 1,

and find the values of ξ and the error probability $P(E)$.

Solution: Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure on the previous page, we easily see that the MAP decision is in favor of H_1 if $|\mathcal{X}| > 0.5$, and hence $x = 1$, $\xi = 0.5$. By inspection, we get that $\pi_0 P_{FA} = 3 \times \frac{1}{12} = \frac{1}{4} = \frac{1}{3} \times \frac{3}{4}$ while $\pi_1 P_{MD} = 2 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{12}\right) = \frac{1}{24} = \frac{2}{3} \times \frac{1}{16}$, that is, $P_{FA} = \frac{3}{4}$, and $P_{MD} = \frac{1}{16}$. $P(E) = \pi_0 P_{FA} + \pi_1 P_{MD} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}$, where $\frac{1}{3}$ is the error probability of the ML rule (with the same *a priori* probabilities) that we found in part (c).

Without using any graphical aids, we have that when $\mathcal{X} = u$ is the observation, the MAP decision rule decides in favor of H_1 if $\Lambda(u) = |u| > \pi_0/\pi_1 = 1/2$. Hence, $\Gamma_1 = \{u : |u| > \frac{1}{2}\}$ and $\Gamma_0 = \{u : |u| < \frac{1}{2}\}$ for the MAP decision rule. Once

again, $x = 1$ while $\xi = \frac{1}{2}$. We get $P_{FA} = \int_{\Gamma_1} f_0(u) du = 2 \int_{\frac{1}{2}}^2 \frac{1}{4} du = \frac{3}{4}$.

$$P_{MD} = \int_{\Gamma_0} f_1(u) du = 2 \int_0^{\frac{1}{2}} \frac{u}{4} du = \frac{u^2}{4} \Big|_0^{\frac{1}{2}} = \frac{1}{16}.$$

Hence, $P(E) = \pi_0 \cdot P_{FA} + \pi_1 \cdot P_{MD} = \frac{1}{3} \times \frac{3}{4} + \frac{2}{3} \times \frac{1}{16} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}$.

- (d) For what range (if any) of values of π_0 , does the MAP decision rule always choose H_0 ? If there are no such values of π_0 , check this box ☐ and leave the answer box below blank. Be sure to justify your answer.

Solution: The MAP decision rule compares the likelihood ratio $\Lambda(u) = |u|$ and decides in favor of H_0 if $\Lambda(u) < \pi_0/\pi_1$. Since $\Lambda(u)$ takes on values in $(0, 2]$, it

will never exceed π_0/π_1 if $\pi_0 > \frac{2}{3}$. Thus, the MAP decision rule always decides in favor of H_0 if $\pi_0 > 2/3$, and achieves average error probability $\bar{P}_e = \pi_1 < 1/3$.

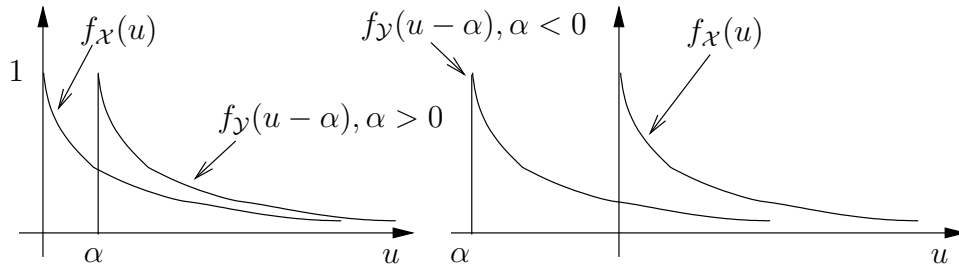
- (e) For what range (if any) of values of π_0 , does the MAP decision rule always choose H_1 ? If there are no such values of π_0 , check this box $\square$ and leave the answer box below blank. Be sure to justify your answer.

Solution On the other hand, there is *no* value of π_0 , $0 < \pi_0 < 1$ for which the MAP rule *always* decide in favor of H_1 , because no matter how large π_1 is, the ratio $\pi_0/\pi_1 > 0$. Thus, for *some* values of $\mathcal{X}$, the likelihood ratio will be smaller than the threshold, and the decision will be in favor of H_0 .

5. $\mathcal{X}$ and $\mathcal{Y}$ are independent exponential random variables with parameter $\lambda = 1$. Find $f_Z(\alpha)$, the pdf of the random variable $Z = \mathcal{X} - \mathcal{Y}$. Be sure to specify the value of $f_Z(\alpha)$ for all real numbers α .

Solution: Since $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables,

$f_Z(\alpha) = \int_{-\infty}^{\infty} f_X(u)f_Y(u - \alpha) du = \int_0^{\infty} \exp(-u)f_Y(u - \alpha) du$ since $f_X(u) = 0$ for $u < 0$. Now, $f_Y(u - \alpha) = 0$ if $u - \alpha < 0$, that is, if $u < \alpha$. As illustrated in the figure below, we have to consider two cases.



- If $\alpha > 0$, then the integrand is 0 for $0 < u < \alpha$, and we get
$$f_Z(\alpha) = \int_{\alpha}^{\infty} \exp(-u) \exp(-u + \alpha) du = \exp(\alpha) \frac{1}{2} \exp(-2u) \Big|_{\alpha}^{\infty} = \frac{1}{2} \exp(-\alpha).$$
- If $\alpha < 0$, then the integrand is nonzero for $0 < u < \infty$, and we get
$$f_Z(\alpha) = \int_0^{\infty} \exp(-u) \exp(-u + \alpha) du = \exp(\alpha) \frac{1}{2} \exp(-2u) \Big|_0^{\infty} = \frac{1}{2} \exp(\alpha).$$

We can combine these two cases and write $f_Z(\alpha) = \frac{1}{2} \exp(-|\alpha|)$, $-\infty < \alpha < \infty$.

6. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables with means 0 and 2 respectively, variances 4 and 16 respectively, and correlation coefficient $\frac{1}{2}$. Let $\mathcal{W} = -\mathcal{X} + b\mathcal{Y}$ and $\mathcal{Z} = 7\mathcal{X} - \mathcal{Y}$ where b is a number whose value you will determine below.

- (a) What value(s) of b minimize $\text{var}(\mathcal{W})$, and what is the minimum value of $\text{var}(\mathcal{W})$?

Solution:

$\text{var}(\mathcal{W}) = \text{var}(\mathcal{X}) + b^2 \text{var}(\mathcal{Y}) - 2b \text{cov}(\mathcal{X}, \mathcal{Y}) = 4 + 16b^2 - 2b \frac{1}{2} 2 \cdot 4 = 16b^2 - 8b + 4$.
This is a quadratic function of b with minimum value 3 at $b = \frac{1}{4}$.

- (b) For what value(s) of b are $\mathcal{W}$ and $\mathcal{Z}$ independent random variables?

Solution: Since $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables, $\mathcal{W}$ and $\mathcal{Z}$ are also jointly Gaussian random variables, and hence independent if $\text{cov}(\mathcal{W}, \mathcal{Z}) = 0$. But,

$$\begin{aligned}\text{cov}(\mathcal{W}, \mathcal{Z}) &= \text{cov}(-\mathcal{X} + b\mathcal{Y}, 7\mathcal{X} - \mathcal{Y}) \\ &= -7 \cdot \text{cov}(\mathcal{X}, \mathcal{X}) + 7b \cdot \text{cov}(\mathcal{Y}, \mathcal{X}) + \text{cov}(\mathcal{X}, \mathcal{Y}) - b \cdot \text{cov}(\mathcal{Y}, \mathcal{Y}) \\ &= -7 \cdot 4 + 7b \cdot 4 + 4 - b \cdot 16 \\ &= 12b - 24\end{aligned}$$

and we see that $\text{cov}(\mathcal{W}, \mathcal{Z}) = 0$ when $b = 2$.